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Transient flow in bounded randomly heterogeneous domains 2. Localization of conditional mean equations and temporal nonlocality effects

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Abstract. In randomly heterogeneous porous media one cannot predict flow behavior with certainty. One can, however, render optimum unbiased predictions of such behavior by means of conditional ensemble mean hydraulic heads and fluxes. We have shown in paper 1 [*Tartakovsky and Neuman*, this issue] that under transient flow, these optimum predictors are governed by nonlocal equations. In particular, the conditional mean flux is generally nonlocal in space-time and therefore non-Darcian. As such, it cannot be associated with an effective hydraulic conductivity except in special cases. Here we explore analytically situations under which localization is possible so that Darcy's law applies in real, Laplace, and/or infinite Fourier transformed spaces, approximately or exactly, with or without conditioning. We show that the corresponding conditional effective hydraulic conductivity tensor is generally nonsymmetric. An alternative to Darcy's law in each case, valid under mean no-flow conditions along Neumann boundaries, is a quasi-Darcian form that includes only a symmetric tensor which, however, does not constitute a bona fide effective hydraulic conductivity. Both lack of symmetry and differences between Darcian and quasi-Darcian forms disappear to first (but not necessarily higher) order of approximation in the (conditional) variance of natural log hydraulic conductivity. We adopt such an approximation to investigate analytically the effect of temporal nonlocality on one- and three-dimensional mean flows in infinite, statistically homogeneous media. Our results show that temporal nonlocality may manifest itself under either monotonic or oscillatory time variations in the mean hydraulic gradient. The effect of temporal nonlocality increases with the variance of log hydraulic conductivity and is more pronounced in one dimension than in three.

1. Introduction

Here, in paper 2, we again consider the problem defined by (1)–(13) in paper 1 [*Tartakovsky and Neuman*, this issue]. The last of these equations expresses the conditional mean flux as (see paper 1 for the definition of all symbols)

$$\langle \mathbf{q}(\mathbf{x}, t)\rangle_c = -\langle K(\mathbf{x})\rangle_c \nabla \langle h(\mathbf{x}, t)\rangle_c + \mathbf{r}_c(\mathbf{x}, t) \tag{1}$$

where $\mathbf{r}_c(\mathbf{x}, t)$ is a residual flux. It was shown that when $\langle Q\rangle \equiv 0$ on Γ_N such that the latter acts as a mean no-flow boundary, the residual flux is given exactly by the explicit expression

$$\mathbf{r}_c(\mathbf{x}, t) = \int_0^t \int_\Omega \mathbf{a}_c(\mathbf{y}, \mathbf{x}, t-\tau)\nabla_y h_c(\mathbf{y}, \tau)\, d\mathbf{y}\, d\tau \tag{2}$$

where

$$\mathbf{a}_c(\mathbf{y}, \mathbf{x}, t-\tau) = \langle K'(\mathbf{y})K'(\mathbf{x})\nabla_x\nabla_y^T G(\mathbf{y}, \mathbf{x}, t-\tau)\rangle_c \tag{3}$$

is a symmetric positive-semidefinite dyadic. Regardless of boundary conditions, one can alternatively express $\mathbf{r}_c$ implicitly as

$$\mathbf{r}_c(\mathbf{x}, t) = \int_0^t \int_\Omega \mathbf{a}_c(\mathbf{y}, \mathbf{x}, t-\tau)\nabla_y \langle h(\mathbf{y}, \tau)\rangle_c\, d\mathbf{y}\, d\tau + \int_0^t \int_\Omega \mathbf{d}_c(\mathbf{y}, \mathbf{x}, t-\tau)\mathbf{r}_c(\mathbf{y}, \tau)\, d\mathbf{y}\, d\tau \tag{4}$$

where

$$\mathbf{d}_c(\mathbf{y}, \mathbf{x}, t-\tau) = \langle K'(\mathbf{x})\nabla_x\nabla_y^T G(\mathbf{y}, \mathbf{x}, t-\tau)\rangle_c \tag{5}$$

is a nonsymmetric dyadic. It is evident from (2)–(5) that the residual flux $\mathbf{r}_c(\mathbf{x}, t)$ is nonlocal and non-Darcian in that it depends on head gradients and residual fluxes at points other than $(\mathbf{x}, t)$. The same is true about the flux predictor $\langle \mathbf{q}(\mathbf{x}, t)\rangle_c$. The purpose of this paper is to explore situations under which the flux predictors $\mathbf{r}_c(\mathbf{x}, t)$ and $\langle \mathbf{q}(\mathbf{x}, t)\rangle_c$ can be localized and to investigate analytically the effect of temporal nonlocality on these quantities.

2. Localization of Conditional Mean Fluxes

Since the flux predictors $\mathbf{r}_c(\mathbf{x}, t)$ and $\langle \mathbf{q}(\mathbf{x}, t)\rangle_c$ are generally nonlocal and non-Darcian, the notion of effective hydraulic conductivity loses meaning in the context of flow prediction by means of conditional ensemble mean quantities. There are, however, a few special situations where localization of the above flux predictors is possible, as we are about to demonstrate.

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For an effective hydraulic conductivity to exist in the strict sense, it is necessary that $\nabla\langle h(\mathbf{x}, t)\rangle_c \equiv$ constant and that $\mathbf{r}_c(\mathbf{x}, t) \equiv$ constant, which never happens under transient conditions unless either $S \equiv 0$ or $t \to \infty$. This will become self evident as we explore below the less restrictive case where $\nabla\langle h(\mathbf{x}, t)\rangle_c$ varies slowly enough in space and time (have negligibly small space and time derivatives) wherever $\mathbf{a}_c(\mathbf{y}, \mathbf{x}, t - \tau) \not\equiv 0$ and where $\mathbf{r}_c(\mathbf{x}, t)$ varies slowly enough in space and time wherever $\mathbf{d}_c(\mathbf{y}, \mathbf{x}, t - \tau) \not\equiv 0$, so that (4) can be approximated by

$$\mathbf{r}_c(\mathbf{x}, t) \approx \mathbf{A}_c(\mathbf{x}, t)\nabla\langle h(\mathbf{x}, t)\rangle_c + \mathbf{B}_c(\mathbf{x}, t)\mathbf{r}_c(\mathbf{x}, t) \quad (6)$$

or

$$\mathbf{r}_c(\mathbf{x}, t) \approx \kappa_c(\mathbf{x}, t)\nabla\langle h(\mathbf{x}, t)\rangle_c \quad (7)$$

where

$$\mathbf{A}_c(\mathbf{x}, t) = \int_0^t \int_\Omega \mathbf{a}_c(\mathbf{y}, \mathbf{x}, t - \tau)\, d\mathbf{y}\, d\tau \quad (8)$$

$$\mathbf{B}_c(\mathbf{x}, t) = \int_0^t \int_\Omega \mathbf{d}_c(\mathbf{y}, \mathbf{x}, t - \tau)\, d\mathbf{y}\, d\tau \quad (9)$$

$$\boldsymbol{\kappa}_c(\mathbf{x}, t) = [\mathsf{I} - \mathbf{B}_c(\mathbf{x}, t)]^{-1}\mathbf{A}_c(\mathbf{x}, t) \quad (10)$$

and I is the identity tensor. Here $\kappa_c(\mathbf{x}, t)$ is a nonsymmetric dyadic unless $\mathbf{B}_c(\mathbf{x}, t) \equiv 0$ in which case it becomes symmetric positive-semidefinite. Substituting (7) into (1) yields an approximate form of Darcy's law for the conditional mean flux,

$$\langle \mathbf{q}(\mathbf{x}, t)\rangle_c \approx -\mathsf{K}_{c,eff}(\mathbf{x}, t)\nabla\langle h(\mathbf{x}, t)\rangle_c \quad (11)$$

where

$$\mathsf{K}_{c,eff}(\mathbf{x}, t) = \langle K(\mathbf{x})\rangle_c \mathsf{I} - \kappa_c(\mathbf{x}, t). \quad (12)$$

Here $\mathsf{K}_{c,\,eff}(\mathbf{x}, t)$ is a space-time varying conditional effective hydraulic conductivity tensor which can be symmetric positive-definite or nonsymmetric, depending on whether $\kappa_c(\mathbf{x}, t)$ is symmetric or nonsymmetric.

An alternative to (6) and (7) which does not require any a priori restrictions on $\mathbf{r}_c(\mathbf{x}, t)$ but does require specifying zero mean flux conditions on Neumann boundaries is obtained by approximating (2) as

$$\mathbf{r}_c(\mathbf{x}, t) \approx \mathbf{A}_c(\mathbf{x}, t)\nabla h_c(\mathbf{x}, t) \quad (13)$$

and (1) in the quasi-Darcian form

$$\langle \mathbf{q}(\mathbf{x}, t)\rangle_c \approx -\langle K(\mathbf{x})\rangle_c \nabla\langle h(\mathbf{x}, t)\rangle_c + \mathbf{A}_c(\mathbf{x}, t)\nabla h_c(\mathbf{x}, t) \quad (14)$$

where the first coefficient is a scalar and the second is a symmetric positive-semidefinite dyadic.

Consider next the case, where $\nabla\langle h(\mathbf{x}, t)\rangle_c$ varies slowly in space but not in time wherever $\mathbf{a}_c(\mathbf{y}, \mathbf{x}, t - \tau) \not\equiv 0$, and $\mathbf{r}_c(\mathbf{x}, t)$ varies slowly in space but not in time wherever $\mathbf{d}_c(\mathbf{y}, \mathbf{x}, t - \tau) \not\equiv 0$, so as to allow approximating (4) by

$$\mathbf{r}_c(\mathbf{x}, t) \approx \int_0^t \boldsymbol{\alpha}_c(\mathbf{x}, t - \tau)\nabla\langle h(\mathbf{x}, \tau)\rangle_c\, d\tau + \int_0^t \boldsymbol{\beta}_c(\mathbf{x}, t - \tau)\mathbf{r}_c(\mathbf{x}, \tau)\, d\tau \quad (15)$$

where

$$\boldsymbol{\alpha}_c(\mathbf{x}, t - \tau) = \int_\Omega \mathbf{a}_c(\mathbf{y}, \mathbf{x}, t - \tau)\, d\mathbf{y} \quad (16)$$

$$\boldsymbol{\beta}_c(\mathbf{x}, t - \tau) = \int_\Omega \mathbf{d}_c(\mathbf{y}, \mathbf{x}, t - \tau)\, d\mathbf{y}. \quad (17)$$

Taking the Laplace transform of (15) yields local residual flux expressions of the form

$$\tilde{\mathbf{r}}_c(\mathbf{x}, \lambda) \approx \tilde{\boldsymbol{\alpha}}_c(\mathbf{x}, \lambda)\nabla\langle \tilde{h}(\mathbf{x}, \lambda)\rangle_c + \tilde{\boldsymbol{\beta}}_c(\mathbf{x}, \lambda)\tilde{\mathbf{r}}_c(\mathbf{x}, \lambda) \quad (18)$$

or

$$\tilde{\mathbf{r}}_c(\mathbf{x}, \lambda) \approx \bar{\kappa}_c(\mathbf{x}, \lambda)\nabla\langle \tilde{h}(\mathbf{x}, \lambda)\rangle_c \quad (19)$$

where the tilde overscript represents Laplace transform, λ is the corresponding transform parameter, and

$$\bar{\kappa}_c(\mathbf{x}, \lambda) = [\mathsf{I} - \tilde{\boldsymbol{\beta}}_c(\mathbf{x}, \lambda)]^{-1}\tilde{\boldsymbol{\alpha}}_c(\mathbf{x}, \lambda). \quad (20)$$

As before, $\bar{\kappa}_c(\mathbf{x}, \lambda)$ is a nonsymmetric dyadic unless $\tilde{\beta}_c(\mathbf{x}, \lambda) \equiv 0$, in which case it becomes symmetric positive semidefinite. Note that $\bar{\kappa}_c(\mathbf{x}, \lambda)$ is not the Laplace transform of $\kappa_c(\mathbf{x}, t)$ in (10). Taking the Laplace transform of (1) and substituting (19) yields an approximate form of Darcy's law for the transformed conditional mean flux,

$$\langle \tilde{\mathbf{q}}(\mathbf{x}, \lambda)\rangle_c \approx -\bar{\mathsf{K}}_{c,eff}(\mathbf{x}, \lambda)\nabla\langle \tilde{h}(\mathbf{x}, \lambda)\rangle_c \quad (21)$$

where

$$\bar{\mathsf{K}}_{c,eff}(\mathbf{x}, \lambda) = \langle K(\mathbf{x})\rangle_c \mathsf{I} - \bar{\kappa}_c(\mathbf{x}, \lambda) \quad (22)$$

is a spatially varying conditional effective hydraulic conductivity tensor in the Laplace domain. The latter can be symmetric positive definite or nonsymmetric, depending on whether $\bar{\kappa}_c(\mathbf{x}, \lambda)$ is symmetric or nonsymmetric. Note that $\bar{\mathsf{K}}_{c,\,eff}(\mathbf{x}, \lambda)$ is not the Laplace transform of $\mathsf{K}_{c,\,eff}(\mathbf{x}, t)$ in (12). Taking the inverse Laplace transform of (21) yields a time-convolution integral for the conditional mean flux,

$$\langle \mathbf{q}(\mathbf{x}, t)\rangle_c \approx -\int_0^t \mathsf{K}^*_{c,eff}(\mathbf{x}, t - \tau)\nabla\langle h(\mathbf{x}, \tau)\rangle_c\, d\tau \quad (23)$$

where $\mathsf{K}^*_{c,\,eff}(\mathbf{x}, t)$ is the inverse Laplace transform of $\bar{\mathsf{K}}_{c,\,eff}(\mathbf{x}, \lambda)$ in (22).

An alternative to (18) and (19) that does not require any a priori restrictions on $\mathbf{r}_c(\mathbf{x}, t)$, but does require specifying zero mean flux conditions on Neumann boundaries, is obtained by approximating (2) as

$$\tilde{\mathbf{r}}_c(\mathbf{x}, \lambda) \approx \tilde{\boldsymbol{\alpha}}_c(\mathbf{x}, \lambda)\nabla\tilde{h}_c(\mathbf{x}, \lambda) \quad (24)$$

and (1) in the quasi-Darcian form

$$\langle \tilde{\mathbf{q}}(\mathbf{x}, \lambda)\rangle_c \approx -\langle K(\mathbf{x})\rangle_c \nabla\langle \tilde{h}(\mathbf{x}, \lambda)\rangle_c + \tilde{\boldsymbol{\alpha}}_c(\mathbf{x}, \lambda)\nabla\tilde{h}_c(\mathbf{x}, \lambda) \quad (25)$$

where the first coefficient is a scalar and the second is a symmetric positive-semidefinite dyadic.

The case where either $\nabla\langle h(\mathbf{x}, t)\rangle_c$ or $\mathbf{r}_c(\mathbf{x}, t)$ vary rapidly in space does not generally lend itself to localization unless one avoids boundary effects and restricts the analysis to statistically homogeneous $K(\mathbf{x})$ fields. We therefore limit consideration here to the special case of unconditional flow in an infinite domain Ω_∞ (where rapid variations in mean gradient and flux may still be caused by mean internal sources) within a homogeneous $K(\mathbf{x})$ field. This case has been studied by *Indelman* [1996] using a different approach than we do. It renders both $\mathbf{a}(\mathbf{y}, \mathbf{x}, t-\tau) \equiv \mathbf{a}(\mathbf{y}-\mathbf{x}, t-\tau)$ and $\mathbf{d}(\mathbf{y}, \mathbf{x}, t-\tau) \equiv \mathbf{d}(\mathbf{y}-\mathbf{x}, t-\tau)$ independent of location, so that $\mathbf{d}$ is now symmetric, and hence (4) can be written in terms of standard space-time convolution integrals as

$$\mathbf{r}(\mathbf{x}, t) = \int_0^t \int_{\Omega_\infty} \mathbf{a}(\mathbf{y}-\mathbf{x}, t-\tau)\nabla\langle h(\mathbf{x}, \tau)\rangle \, d\mathbf{y}\, d\tau + \int_0^t \int_{\Omega_\infty} \mathbf{d}(\mathbf{y}-\mathbf{x}, t-\tau)\mathbf{r}(\mathbf{x}, \tau)\, d\mathbf{y}\, d\tau. \tag{26}$$

The fact that $\mathbf{d}$ is independent of location and is symmetric follows from (5) upon recognizing that $K'(\mathbf{x})$ now has the same statistical properties as $K'(\mathbf{y})$. Equation (26) lends itself to double Laplace and infinite Fourier transformation and yields the following exact local expressions for residual flux in Laplace-Fourier space:

$$\hat{\tilde{\mathbf{r}}}(\boldsymbol{\xi}, \lambda) = \hat{\tilde{\mathbf{a}}}(\boldsymbol{\xi}, \lambda)\nabla\langle\hat{\tilde{h}}(\boldsymbol{\xi}, \lambda)\rangle + \hat{\tilde{\mathbf{d}}}(\boldsymbol{\xi}, \lambda)\hat{\tilde{\mathbf{r}}}(\boldsymbol{\xi}, \lambda) \tag{27}$$

or

$$\hat{\tilde{\mathbf{r}}}(\boldsymbol{\xi}, \lambda) = \bar{\tilde{\boldsymbol{\kappa}}}(\boldsymbol{\xi}, \lambda)\nabla\langle\hat{\tilde{h}}(\boldsymbol{\xi}, \lambda)\rangle \tag{28}$$

where the combination of circumflex and tilde accents represents double transform, $\boldsymbol{\xi}$ is a wave number vector (representing spatial frequencies), and

$$\bar{\tilde{\boldsymbol{\kappa}}}(\boldsymbol{\xi}, \lambda) = [\mathsf{I} - \hat{\tilde{\mathbf{d}}}(\boldsymbol{\xi}, \lambda)]^{-1}\hat{\tilde{\mathbf{a}}}(\boldsymbol{\xi}, \lambda). \tag{29}$$

Here $\bar{\tilde{\boldsymbol{\kappa}}}(\boldsymbol{\xi}, \lambda)$ is a symmetric dyadic which, however, does not constitute a double transform of $\boldsymbol{\kappa}(\mathbf{x}, t)$ in (10). Taking the double transform of (1) and substituting (28) yields the following exact form of Darcy's law for the double transformed unconditional mean flux:

$$\langle\hat{\tilde{\mathbf{q}}}(\boldsymbol{\xi}, \lambda)\rangle = -\bar{\tilde{\mathsf{K}}}_{eff}(\boldsymbol{\xi}, \lambda)\nabla\langle\hat{\tilde{h}}(\boldsymbol{\xi}, \lambda)\rangle \tag{30}$$

where

$$\bar{\tilde{\mathsf{K}}}_{eff}(\boldsymbol{\xi}, \lambda) = \langle K\rangle\mathsf{I} - \bar{\tilde{\kappa}}(\boldsymbol{\xi}, \lambda) \tag{31}$$

is a symmetric effective hydraulic conductivity tensor in Laplace-Fourier space. The latter is not the Laplace-Fourier transform of $\mathsf{K}_{eff}(\mathbf{x}, t)$ in (12). Taking the inverse double transform of (30) yields a space-time convolution integral for the unconditional mean flux,

$$\langle\mathbf{q}(\mathbf{x}, t)\rangle = -\int_0^t \int_{\Omega_\infty} \mathsf{K}^{**}_{eff}(\mathbf{x}-\boldsymbol{\chi}, t-\tau)\nabla\langle h(\boldsymbol{\chi}, \tau)\rangle \, d\boldsymbol{\chi}\, d\tau \tag{32}$$

where $\mathsf{K}^{**}_{eff}(\mathbf{x}, t)$ is the inverse Laplace-Fourier transform of $\bar{\tilde{\mathsf{K}}}_{eff}(\boldsymbol{\xi}, \lambda)$ in (31). Our (30)–(32) are analogous, respectively, to (18)–(19) and (21) of *Indelman* [1996]. However, our method of derivation is different, and therefore our (29) looks different than Indelman's (19), though the two expressions must be identical in principle.

An alternative to (28)–(30) which does not require any a priori restrictions on $\mathbf{r}(\mathbf{x}, t)$ is obtained exactly from (2) as

$$\hat{\tilde{\mathbf{r}}}(\boldsymbol{\xi}, \lambda) = \hat{\tilde{\mathbf{a}}}(\boldsymbol{\xi}, \lambda)\nabla\hat{\tilde{h}}(\boldsymbol{\xi}, \lambda) \tag{33}$$

so that (1) takes the exact quasi-Darcian form

$$\langle\hat{\tilde{\mathbf{q}}}(\boldsymbol{\xi}, \lambda)\rangle = -\langle K\rangle\nabla\langle\hat{\tilde{h}}(\boldsymbol{\xi}, \lambda)\rangle + \hat{\tilde{\mathbf{a}}}(\boldsymbol{\xi}, \lambda)\nabla\hat{\tilde{h}}_c(\boldsymbol{\xi}, \lambda) \tag{34}$$

where the first coefficient is a scalar constant and the second is a symmetric positive-semidefinite dyadic.

3. First-Order Approximations

To render the above formal mean flow expressions workable, we expand them below in a small parameter σ_Y representing a measure of the standard deviation of $Y'(\mathbf{x}) = Y(\mathbf{x}) - \langle Y(\mathbf{x})\rangle_c$ where $Y(\mathbf{x}) = \ln K(\mathbf{x})$; this nominally limits our approximation either to mildly heterogeneous media or to well-conditioned strongly heterogeneous, media with $\sigma_Y < 1$. It has been shown in part 1 [*Tartakovsky and Neuman*, this issue] that to zeroth and first orders in σ_Y, the residual flux is identically equal to zero. To second order in σ_Y (first order in σ_Y^2) it is given uniquely, regardless of boundary conditions, by

$$\mathbf{r}_c^{(2)}(\mathbf{x}, t) = \int_0^t \int_{\Omega} \mathbf{a}_c^{(2)}(\mathbf{y}, \mathbf{x}, t-\tau)\nabla_y h_c^{(0)}(\mathbf{y}, \tau)\, d\mathbf{y}\, d\tau \tag{35}$$

where $^{(i)}$ denotes terms that contain only ith powers of σ_Y,

$$\mathbf{a}_c^{(2)}(\mathbf{y}, \mathbf{x}, t-\tau) = K_G(\mathbf{x})K_G(\mathbf{y})\langle Y'(\mathbf{x})Y'(\mathbf{y})\rangle_c \nabla_x\nabla_y^T G_c^{(0)}(\mathbf{y}, \mathbf{x}, t-\tau) \tag{36}$$

$K_G(\mathbf{x}) = \exp\langle Y(\mathbf{x})\rangle_c$ is the conditional geometric mean of $Y(\mathbf{x})$, and $G_c^{(0)}$ is a zeroth-order conditional Green's function. Hence to second (but not necessarily higher) order in σ_Y, the nonsymmetric kernel $\mathbf{d}_c$ drops out from the residual flux in (4) and the latter equation is identical to (2), which in turn is now valid for arbitrary boundary conditions. Likewise, there is no difference between $h_c^{(0)}(\mathbf{y}, \tau)$ and $\langle h^{(0)}(\mathbf{y}, \tau)\rangle_c$, both of which satisfy

$$S(\mathbf{x})\frac{\partial h_c^{(0)}(\mathbf{x}, t)}{\partial t} = -\nabla\cdot\mathbf{q}_c^{(0)}(\mathbf{x}, t) + \langle f(\mathbf{x}, t)\rangle \qquad \mathbf{q}_c^{(0)}(\mathbf{x}, t) = -K_G(\mathbf{x})\nabla h_c^{(0)}(\mathbf{x}, t) \tag{37}$$

subject to the original mean initial and boundary conditions

$$h_c^{(0)}(\mathbf{x}, 0) = \langle H_0(\mathbf{x})\rangle \qquad \mathbf{x}\in\Omega \tag{38}$$

$$h_c^{(0)}(\mathbf{x}, t) = \langle H(\mathbf{x}, t)\rangle \qquad \mathbf{x}\in\Gamma_D \tag{39}$$

$$-\mathbf{q}_c^{(0)}(\mathbf{x}, t)\cdot\mathbf{n}(\mathbf{x}) = \langle Q(\mathbf{x}, t)\rangle \qquad \mathbf{x}\in\Gamma_N \tag{40}$$

On the other hand, $\langle h(\mathbf{x}, t)\rangle_c^{(1)} \equiv 0$, $\langle\mathbf{q}(\mathbf{x}, t)\rangle_c^{(1)} \equiv 0$, and $\langle h(\mathbf{x}, t)\rangle_c^{(2)}$ satisfies

$$S(\mathbf{x})\frac{\partial\langle h^{(2)}(\mathbf{x}, t)\rangle_c}{\partial t} = \nabla\cdot\langle\mathbf{q}^{(2)}(\mathbf{x}, t)\rangle_c \tag{41}$$

subject to homogeneous mean initial and boundary conditions with flux given by

$$\langle\mathbf{q}^{(2)}(\mathbf{x}, t)\rangle_c = -K_G(\mathbf{x})\left[\nabla\langle h^{(2)}(\mathbf{x}, t)\rangle_c + \frac{\sigma_Y^2(\mathbf{x})}{2}\nabla h_c^{(0)}(\mathbf{x}, t)\right] + \mathbf{r}_c^{(2)}(\mathbf{x}, t) \tag{42}$$

where $\sigma_Y^2(\mathbf{x}) = \langle Y'^2(\mathbf{x})\rangle_c$.

Let $^{[2]}$ denote the sum of all terms containing σ_Y raised up to second power. It then follows from (35), (36), (37), and (42) that when conditional mean gradient and flux vary slowly in both space and time, (6)–(14) reduce to

$$\mathbf{r}_c^{[2]}(\mathbf{x}, t) \approx \kappa_c^{[2]}(\mathbf{x}, t)\nabla\langle h^{(0)}(\mathbf{x}, t)\rangle_c \tag{43}$$

where

$$\kappa_c^{[2]}(\mathbf{x}, t) = K_G(\mathbf{x})\int_0^t \int_\Omega K_G(\mathbf{y})\mathsf{C}_c(\mathbf{y}, \mathbf{x}) \nabla_x\nabla_y^T G^{(0)}(\mathbf{y}, \mathbf{x}, t-\tau)\, d\mathbf{y}\, d\tau \tag{44}$$

is now symmetric positive semidefinite and $\mathsf{C}_c(\mathbf{y}, \mathbf{x}) = \langle Y'(\mathbf{x})Y'(\mathbf{y})\rangle_c$ is the conditional spatial autocovariance of Y, and

$$\langle \mathbf{q}^{[2]}(\mathbf{x}, t)\rangle_c \approx -K_G(\mathbf{x})\nabla\langle h^{(2)}(\mathbf{x}, t)\rangle_c - \mathsf{K}_{c,\mathit{eff}}^{[2]}(\mathbf{x}, t)\nabla h_c^{(0)}(\mathbf{x}, t) \tag{45}$$

where

$$\mathsf{K}_{c,\mathit{eff}}^{[2]}(\mathbf{x}, t) = K_G(\mathbf{x})\left(1 + \frac{\sigma_Y^2(\mathbf{x})}{2}\right)\mathsf{I} - \kappa_c^{[2]}(\mathbf{x}, t) \tag{46}$$

is symmetric positive definite.

By the same token, when conditional mean gradient and flux vary slowly in space but not in time, (19)–(22) and (23)–(25) reduce to

$$\bar{\mathbf{r}}_c^{[2]}(\mathbf{x}, \lambda) \approx \bar{\kappa}_c^{[2]}(\mathbf{x}, \lambda)\nabla\langle \bar{h}^{(0)}(\mathbf{x}, \lambda)\rangle_c \tag{47}$$

where

$$\bar{\kappa}_c^{[2]}(\mathbf{x}, \lambda) = K_G(\mathbf{x})\int_\Omega K_G(\mathbf{y})\mathsf{C}_c(\mathbf{y}, \mathbf{x})\nabla_x\nabla_y^T\bar{G}_c^{(0)}(\mathbf{y}, \mathbf{x}, \lambda)\, d\mathbf{y} \tag{48}$$

is symmetric positive semidefinite and

$$\langle \bar{\mathbf{q}}^{[2]}(\mathbf{x}, \lambda)\rangle_c \approx -K_G(\mathbf{x})\nabla\langle \bar{h}^{(2)}(\mathbf{x}, \lambda)\rangle_c - \bar{\mathsf{K}}_{c,\mathit{eff}}^{[2]}(\mathbf{x}, \lambda)\nabla \bar{h}_c^{(0)}(\mathbf{x}, \lambda) \tag{49}$$

where

$$\bar{\mathsf{K}}_{c,\mathit{eff}}^{[2]}(\mathbf{x}, \lambda) = K_G(\mathbf{x})\left(1 + \frac{\sigma_Y^2(\mathbf{x})}{2}\right)\mathsf{I} - \bar{\kappa}_c^{[2]}(\mathbf{x}, \lambda) \tag{50}$$

is symmetric positive definite.

When mean gradient and flux vary rapidly in both space and time, localization is restricted to unconditional flow in an infinite domain (where rapid variations in mean gradient and flux may now be caused by mean internal sorces) within which $K(\mathbf{x})$ is statistically homogeneous. Then (28)–(31) and (33) and (34) reduce to

$$\hat{\bar{\mathbf{r}}}^{[2]}(\boldsymbol{\xi}, \lambda) = \bar{\bar{\kappa}}^{[2]}(\boldsymbol{\xi}, \lambda)\nabla\langle \hat{\bar{h}}^{(0)}(\boldsymbol{\xi}, \lambda)\rangle \tag{51}$$

where

$$\bar{\bar{\kappa}}^{[2]}(\boldsymbol{\xi}, \lambda) = \hat{\bar{\mathbf{a}}}^{[2]}(\boldsymbol{\xi}, \lambda) \tag{52}$$

is symmetric positive semidefinite and

$$\langle \hat{\bar{\mathbf{q}}}^{[2]}(\boldsymbol{\xi}, \lambda)\rangle = -K_G\nabla\langle \hat{\bar{h}}^{(2)}(\boldsymbol{\xi}, \lambda)\rangle - \bar{\bar{\mathsf{K}}}_{\mathit{eff}}^{[2]}(\boldsymbol{\xi}, \lambda)\nabla\hat{\bar{h}}^{(0)}(\boldsymbol{\xi}, \lambda) \tag{53}$$

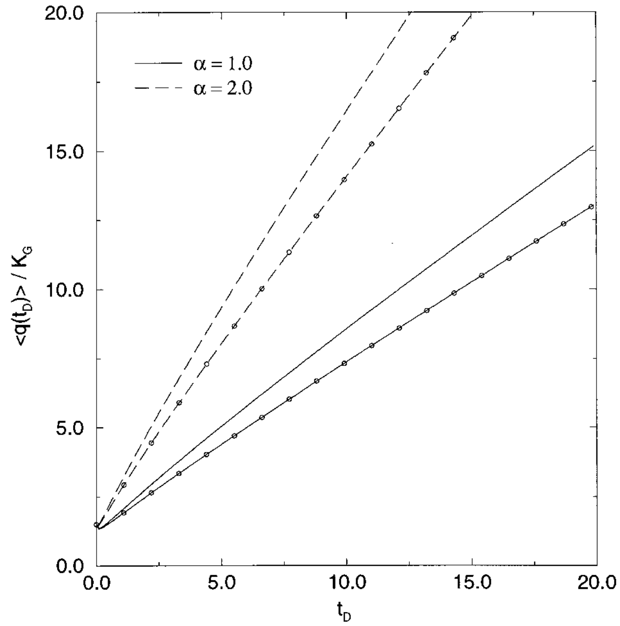

Figure 1. Normalized nonlocal and localized (circles) mean flux in one dimension versus dimensionless time for $J(t_D) = 1 + \alpha t_D$ and two values of α when $\sigma_Y^2 = 1$.

where

$$\bar{\bar{\mathsf{K}}}_{\mathit{eff}}^{[2]}(\boldsymbol{\xi}, \lambda) = K_G\left(1 + \frac{\sigma_Y^2}{2}\right)\mathsf{I} - \bar{\bar{\kappa}}^{[2]}(\boldsymbol{\xi}, \lambda) \tag{54}$$

is symmetric positive definite.

4. Effective Hydraulic Conductivity in Laplace Space

Since time localization is always possible in Laplace space, it is of general interest to derive analytical expressions for effective hydraulic conductivity in this space. We do so to first order in σ_Y^2 for the relatively simple cases of one- and three-dimensional unconditional flows in infinite, statistically homogeneous hydraulic conductivity fields under a spatially uniform but time-varying mean hydraulic gradient $\mathbf{J}(t)$. Then $\langle h^{(2)}(\mathbf{x}, t)\rangle \equiv 0$ and (47)–(50) simplify to

$$\bar{\mathbf{r}}^{[2]}(\lambda) \approx \bar{\kappa}^{[2]}(\lambda)\mathbf{J}(\lambda) \tag{55}$$

where

$$\bar{\kappa}^{[2]}(\lambda) = K_G\int_{\Omega_\infty}\mathsf{C}(\mathbf{y}-\mathbf{x}) \nabla_x\nabla_y^T\bar{G}_K^{(0)}(\mathbf{y}-\mathbf{x}, \lambda)\, d\mathbf{y} \qquad \mathbf{x} \text{ arbitrary} \tag{56}$$

$G_K^{(0)} = K_G G^{(0)}$, and

$$\langle \bar{\mathbf{q}}^{[2]}(\lambda)\rangle = -\bar{\mathsf{K}}_{\mathit{eff}}^{[2]}\bar{\mathbf{J}}(\lambda) \tag{57}$$

where

$$\bar{\mathsf{K}}_{\mathit{eff}}^{[2]}(\lambda) = K_G\left(1 + \frac{\sigma_Y^2}{2}\right)\mathsf{I} - \bar{\kappa}^{[2]}(\lambda). \tag{58}$$

The inverse Laplace transform of (57) and (58) is

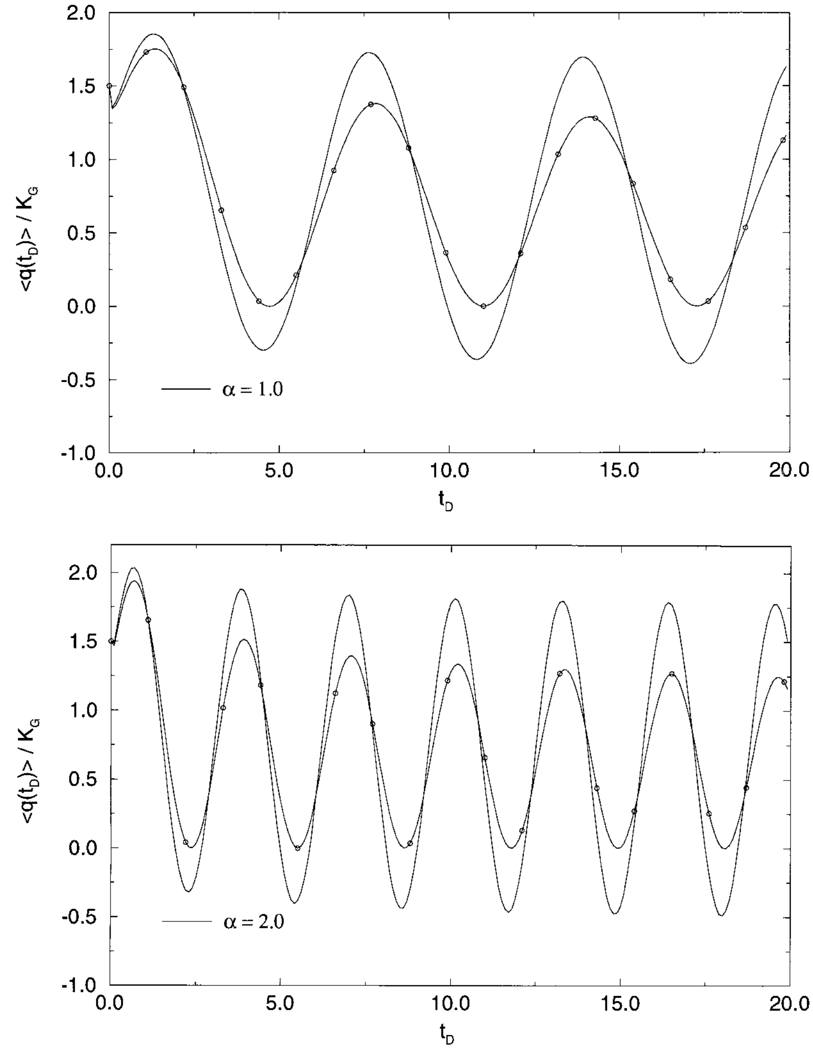

Figure 2. Normalized nonlocal and localized (circles) mean flux in one dimension versus dimensionless time for $J(t_D) = 1 + \sin(\alpha t_D)$ and two values of α when $\sigma_Y^2 = 1$.

$$\langle \mathbf{q}^{[2]}(t) \rangle = -K_G\left(1 + \frac{\sigma_Y^2}{2}\right)\mathbf{J}(t) + \int_0^t \kappa^{*[2]}(t-\tau)\mathbf{J}(\tau)\,d\tau \tag{59}$$

where

$$\kappa^{*[2]}(t) = K_G \int_\Omega \mathsf{C}(\mathbf{y}-\mathbf{x}) \nabla_x \nabla_y^T G_K^{(0)}(\mathbf{y}-\mathbf{x}, t)\,d\mathbf{y} \qquad \mathbf{x} \text{ arbitrary} \tag{60}$$

is the inverse Laplace transform of (56).

For comparison purposes we also consider the special case where $\mathbf{J}(t)$ varies slowly in time.

Then (59) simplifies to

$$\langle \mathbf{q}^{[2]}(t) \rangle \approx -\mathsf{K}_{eff}^{[2]}(t)\mathbf{J}(t) \tag{61}$$

where

$$\mathsf{K}_{eff}^{[2]}(t) = K_G\left(1 + \frac{\sigma_Y^2}{2}\right)\mathsf{I} - \int_0^t \kappa^{*[2]}(\tau)\,d\tau \tag{62}$$

and thus

$$\kappa^{*[2]}(t) = -\frac{d\mathsf{K}_{eff}^{[2]}(t)}{dt}. \tag{63}$$

To explore the effect of temporal nonlocality on mean flow behavior we consider below the special case where S is constant and where $Y(\mathbf{x})$ is homogeneous Gaussian with constant mean K_G, variance σ_Y^2, integral scale I, and isotropic exponential spatial autocovariance. The case where $J(t)$ varies slowly in time has been analyzed by *Dagan* [1982]. His corresponding expressions for one and three dimensions read, respectively,

$$\frac{\langle q^{[2]}(t_D) \rangle}{K_G} = \left\{1 + \sigma_Y^2\left[-\frac{1}{2} + e^{t_D}\operatorname{erfc}\sqrt{t_D}\right]\right\}J(t_D) \tag{64}$$

$$\frac{\langle q^{[2]}(t_D) \rangle}{K_G} = \left\{1 + \frac{\sigma_Y^2}{3}\left[\frac{1}{2} - 2\sqrt{\frac{t_D}{\pi}} + (1 + 2t_D)e^{t_D}\operatorname{erfc}\sqrt{t_D}\right]\right\}J(t_D) \tag{65}$$

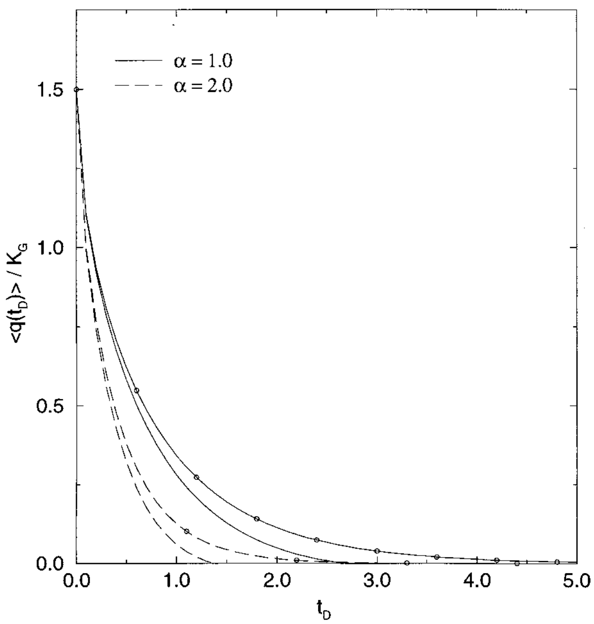

Figure 3. Normalized nonlocal and localized (circles) mean flux in one dimension versus dimensionless time for $J(t_D) = 1 + \exp(-\alpha t_D)$ and two values of α when $\sigma_Y^2 = 1$.

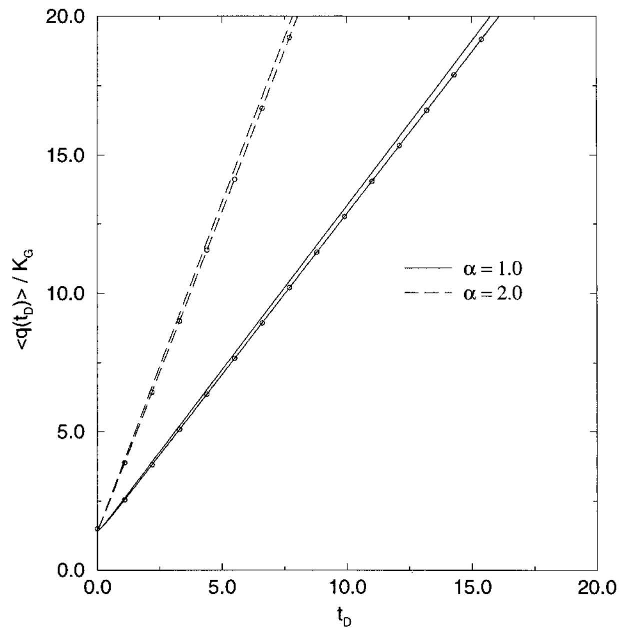

Figure 4. Normalized nonlocal and localized (circles) mean flux in three dimensions versus dimensionless time for $J(t_D) = 1 + \alpha t_D$ and two values of α when $\sigma_Y^2 = 1$.

where $t_D = K_G t/SI^2$ is dimensionless time and $J(t_D) = \|-\mathbf{J}(t_D)\|$ is the magnitude (Euclidean norm) of the negative mean hydraulic gradient. By applying (63) followed by (59) to (64) and (65), we obtain the following corresponding expressions for the case where $J(t)$ varies arbitrarily with time:

$$\frac{\langle q^{[2]}(t_D)\rangle}{K_G} = \left(1 + \frac{\sigma_Y^2}{2}\right) J(t_D) - \sigma_Y^2 \int_0^{t_D} \left[\frac{1}{\sqrt{\pi(t_D - \tau_D)}} - e^{t_D - \tau_D} \operatorname{erfc} \sqrt{t_D - \tau_D}\right] J(\tau_D)\, d\tau_D \qquad (66)$$

in one dimension and

$$\frac{\langle q^{[2]}(t_D)\rangle}{K_G} = \left(1 + \frac{\sigma_Y^2}{2}\right) J(t_D) - \frac{\sigma_Y^2}{\sqrt{\pi}} \int_0^{t_D} \left\{\frac{2}{3\sqrt{(t_D - \tau_D)}} + \frac{2\sqrt{t_D - \tau_D}}{3} - \sqrt{\pi}\left[1 + \frac{2(t_D - \tau_D)}{3}\right] e^{t_D - \tau_D} \operatorname{erfc} \sqrt{t_D - \tau_D}\right\} J(\tau_D)\, d\tau_D \qquad (67)$$

in three dimensions.

We evaluated (64)–(67) with $J(t_D) = 1 + \alpha t_D$, $J(t_D) = 1 + \sin(\alpha t_D)$ and $J(t_D) = 1 + \exp(-\alpha t_D)$ for several values of α and σ_Y^2. We found that when $\sigma_Y^2 = 0.1$, there is very little difference between localized and time-nonlocal behaviors as given, respectively, by (64) and (65), and (66) and (67). To illustrate the nonlocality effects, we recall from *Hsu and Neuman* [1997, p. 633] that although perturbation expansions are not guaranteed to work for $\sigma_Y^2 \geq 1$, they often do work quite well for $1 \leq \sigma_Y^2 \leq 2$. Since nonlocality effects are more easily discernible within this latter range than when $\sigma_Y^2 < 1$, we elect to illustrate them for $\sigma_Y^2 = 1$. Figures 1–3 show how mean flux (normalized by K_G) varies with t_D and α for each of the above three $J(t_D)$ functions in one dimension when $\sigma_Y^2 = 1$, and Figures 4–6 show the same in three dimensions. The difference between time-nonlocal and time-localized behaviors is now evident and is seen to be more pronounced in one dimension than in three. We also see that temporal nonlocality manifests itself not only when the mean hydraulic gradient oscillates, but also when it increases or decreases monotonically, with time.

5. Conclusions

We are thus led to the following conclusions:

1. In randomly heterogeneous porous media one cannot predict flow behavior with certainty. One can, however, render optimum unbiased predictions of such behavior by means of conditional ensemble mean hydraulic heads and fluxes. Under transient flow these optimum predictors are governed by nonlocal (integrodifferential) equations. In particular, the conditional mean flux is generally nonlocal in space-time and therefore non-Darcian. As such, it cannot be associated with an effective hydraulic conductivity except in special cases. We have explored analytically situations under which localization of mean transient flow is possible so that Darcy's law applies in real, Laplace, and/or infinite Fourier transformed spaces, approximately or exactly, with or without conditioning.

2. When the conditional mean hydraulic gradient and flux vary slowly in both space and time, approximate localization is possible in real space-time. When these quantities vary slowly in space but not in time, such localization is possible in Laplace space. Localization is not possible when the conditional mean gradient and flux vary rapidly in space. However, it becomes possible exactly in Laplace-Fourier space when one restricts

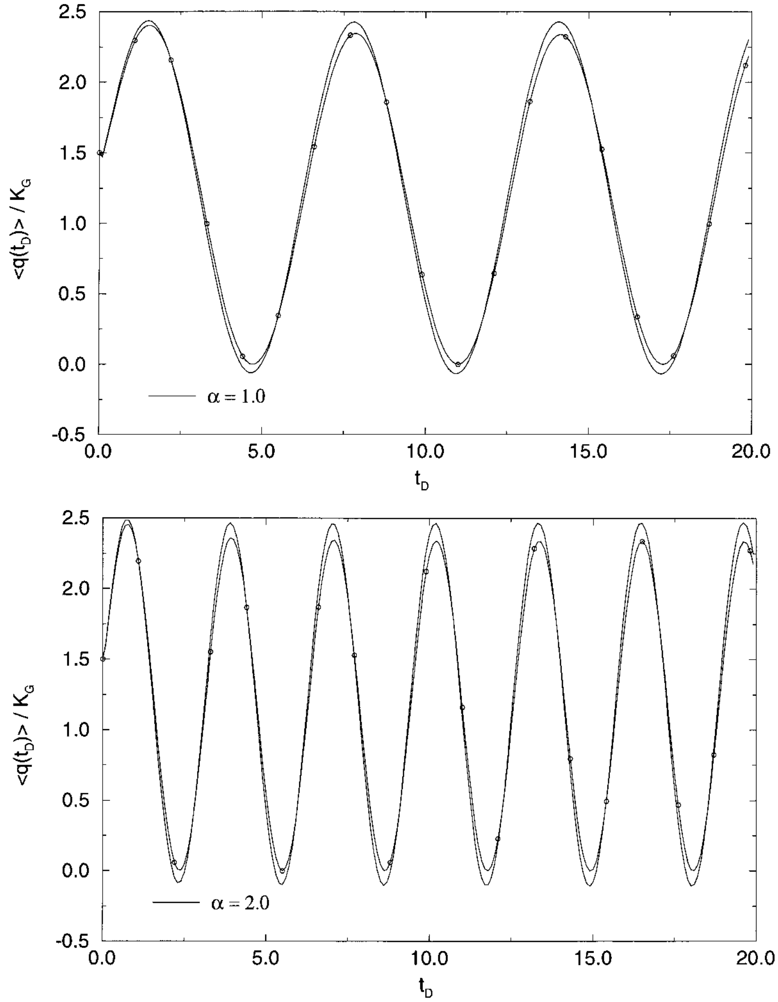

Figure 5. Normalized nonlocal and localized (circles) mean flux in three dimensions versus dimensionless time for $J(t_D) = 1 + \sin(\alpha t_D)$ and two values of α when $\sigma_Y^2 = 1$.

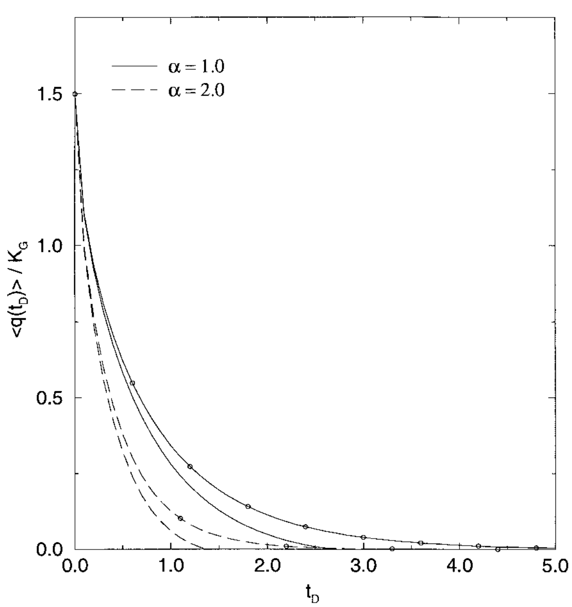

Figure 6. Normalized nonlocal and localized (circles) mean flux in three dimensions versus dimensionless time for $J(t_D) = 1 + \exp(-\alpha t_D)$ and two values of α when $\sigma_Y^2 = 1$.

consideration to unconditional flow in infinite domains within which the hydraulic conductivity is statistically homogeneous. The latter unconditional case has been the focus of an earlier study by *Indelman* [1996].

3. Regardless of whether localization is accomplished in real or transformed space, the corresponding conditional effective hydraulic conductivity tensor is generally nonsymmetric. It becomes symmetric positive definite in the unconditional case where flow takes place in an infinite domain and the hydraulic conductivity is statistically homogeneous. An alternative to Darcy's law in each space, valid under mean no-flow conditions along Neumann boundaries, is a quasi-Darcian form that includes only a symmetric tensor which, however, does not constitute a bona fide effective hydraulic conductivity.

4. Both lack of symmetry and differences between Darcian and quasi-Darcian forms disappear to first (but not necessarily higher) order of approximation in the conditional variance of natural log hydraulic conductivity. Such a first-order approximation is valid either in mildly heterogeneous or in well-conditioned strongly heterogeneous media.

5. We adopted a first-order approximation to investigate analytically the effect of temporal nonlocality on one- and three-dimensional mean flows in infinite, statistically homogeneous media. Our results have shown that temporal nonlocality

may manifest itself under either monotonic or oscillatory time variations in the mean hydraulic gradient. Its effect increases with the variance of log hydraulic conductivity and is more pronounced in one dimension than in three.

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