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Unsaturated hydraulic conductivity function based on a soil fragmentation process

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Abstract. We present a new two-parameter expression for relative hydraulic conductivity (RHC) of partially saturated soils. It is based on the premise of *Assouline et al.* [1998] that soil structures evolve from a uniform random fragmentation process. This assumption allows us to derive hydraulic properties of soils (water retention curves and unsaturated hydraulic conductivity) from primary properties, such as pore geometry and soil structure. We tested our RHC expression against different soil types and found that it fits data better than the widely used models of *Brooks and Corey* [1964] and *van Genuchten* [1980].

1. Introduction

Most studies describing movement of water through partially saturated porous media are based on Richards equation. To solve this equation, a water retention curve (WRC), $\psi(\theta)$, which describes the relationship between capillary head, ψ, and moisture content, θ, is required. Additionally, the dependence of relative hydraulic conductivity (RHC) on moisture content, $K_r(\theta)$, needs to be specified.

WRCs are typically determined from laboratory measurements and a subsequent fitting of a particular functional form of $\psi(\theta)$ to data. Among a variety of models the WRCs proposed by *Brooks and Corey* [1964] and *van Genuchten* [1980] are most popular. Given the saturated, θ_s, and residual, θ_r, water contents, the effective saturation S_e is defined by $S_e = (\theta - \theta_r)/(\theta_s - \theta_r)$. The Brooks and Corey model represents S_e as a power function of ψ,

$$
\begin{aligned}
S_e &= \left(\frac{\psi}{\psi_c}\right)^{-\lambda} \qquad \psi < \psi_c, \\
S_e &= 1 \qquad \psi \geq \psi_c,
\end{aligned}
\tag{1}
$$

where ψ_c and λ are the fitting parameters. The main shortcomings of (1) are, first, the absence of an inflection point which might cause discrepancies with field-measured data [*Milly*, 1987] and, second, the sharp discontinuity of the derivative at ψ_c. The presence of an inflection point in the van Genuchten WRC,

$$
S_e(\psi) = \left[1 + \left(\frac{\psi}{\psi_1}\right)^{\alpha}\right]^{-\beta}, \tag{2}
$$

where ψ_1, α, and β are the fitting parameters, allows for better performance, particularly near saturation [*van Genuchten and Nielsen*, 1985]. Conversely, *Nimmo* [1991] and *Ross et al.* [1991] found that the van Genuchten WRC (2) often performs poorly at low water contents. Additionally, the Brooks and Corey and van Genuchten WRCs are largely empirical and disconnected from basic soil properties, such as pore geometry and adsorption [*Hillel*, 1980, p. 149].

Table 1. Soils and the Saturated, θ_s, and Residual, θ_r, Water Contents Used in our Analysis[a]

Soil Type	Catalog Number	θ_s, m^3/m^3	θ_r, m^3/m^3
Rubicon sandy loam	3501	0.381	0.166
Pachappa loam	3403	0.456	0.075
Beit Netofa clay	4118	0.460	0.242
Pachappa fine sandy clay	3503	0.334	0.049
Amarillo silty clay loam	3002	0.455	0.110
Sable de Riviere	1006	0.342	0.075

[a]Data are taken from the catalog of *Mualem* [1974].

Recently, *Assouline et al.* [1998] derived a WRC from basic soil properties. Starting from the premise that soil structure evolves from a uniform random fragmentation process, they arrived at a two-parameter model for WRC,

$$
S_e(\psi) = 1 - \exp\left[-\xi\left(\frac{1}{|\psi|} - \frac{1}{|\psi_L|}\right)^{\eta}\right], \tag{3}
$$

where $0 \leq |\psi| \leq |\psi_L|$, ψ_L is the capillary head at the wilting point, and ξ and η are the fitting parameters. The units of ξ correspond to those of ψ^{η}. In the subsequent analysis we take $\psi_L = -158.5$ m so that (3) becomes a two-parameter model. Analyzing 12 data sets, which represent a wide range of soil textures, *Assouline et al.* [1998] demonstrated that (3) fits data better and is suitable for a wider range of water contents than the WRCs proposed by *Brooks and Corey* [1964] and *van Genuchten* [1980].

Once the WRC has been chosen, RHC can be obtained by a variety of methods [*Mualem*, 1986]. Among those, *Mualem*'s [1976] model,

$$
K_r(S_e) = \sqrt{S_e}\left[\frac{\int_0^{S_e} \psi^{-1}\, dS_e}{\int_0^1 \psi^{-1}\, dS_e}\right]^2, \tag{4}
$$

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Table 2. Fitting Parameters for the Brooks and Corey (1), van Genuchten (2), and New (3) models

Soil Type	ξ in Equation (3)	η in Equation (3)	λ in Equation (1)	ψ_c in Equation (1)	β in Equation (2)	ψ_1 in Equation (2)
Rubicon sandy loam	0.506	3.943	2.85	0.70	4.419	0.88
Pachappa loam	1.973	1.163	0.87	1.00	2.195	1.65
Beit Netofa clay	3.0	1.453	0.11	1.30	1.162	4.35
Pachappa fine sandy clay	1.025	0.919	0.60	0.40	1.860	0.80
Amarillo silty clay loam	0.880	2.840	2.00	0.73	4.510	1.015
Sable de Riviere	7.8×10^{-3}	4.480	2.92	0.165	6.713	0.22

is most popular. Closed-form analytical expressions for $K_r(S_e)$ exist for some analytical $\psi(S_e)$ functions. In particular, substituting (1) and (2) into (4) gives rise to the Brooks and Corey,

$$K_r(S_e) = S_e^{(2+2.5\lambda)/\lambda}, \tag{5}$$

and the van Genuchten,

$$K_r(S_e) = \sqrt{S_e}\,[1 - S_e^{1/\beta})^{\beta}]^2, \tag{6}$$

expressions for RHC. The Brooks and Corey model (5) was found to be very sensitive to the choice of the parameter λ. In turn, this choice is very sensitive to a procedure used to fit (1) to the WRC data and to the resulting values of ψ_c and θ_r. A simple, closed-form expression for the van Genuchten model (6) is derived by setting $\beta = 1 - \alpha^{-1}$ in (2) which restricts its versatility. A more general, but also much more complex, RHC expression which preserves the two fitting parameters, α and β, was derived by *van Genuchten and Nielsen* [1985]. However, because of its simplicity, (6) is used most often.

In this paper we derive an expression for RHC based on the fragmentation process WRC (3) and Mualem's model (4). We further test this expression against data collected from four soils and compare it with the Brooks and Corey (5) and van Genuchten (6) models.

2. Relative Hydraulic Conductivity Function

The WRC model of *Assouline et al*. [1998] is based on the premise that the particle volume distribution in natural soils results from a series of sequential fragmentations. These fragmentations are caused by the cycles of wetting and drying; physical, chemical and biological processes; and cultivation practices. *Assouline et al*. [1998] assumed that the fragmentation process is uniform and random and that probability for a particle fragmentation is proportional to its volume. These assumptions resulted in a probability distribution function of soil particles which tends asymptotically to an exponential distribution. The latter is a particular case of the Weibull distribution [*Tenchov and Yanev*, 1986]. *Assouline and Rouault* [1997] and *Rouault and Assouline* [1998] established a power relationship between particle volume and pore volume. This relationship results in a pore volume probability distribution being the general Weibull distribution. Substituting the capillary law into the pore volume probability distribution leads to the WRC model (3).

The corresponding RHC model is derived by substituting (3) into (4). Evaluating the integrals yields (Appendix A)

$$K_r(S_e) = \sqrt{S_e}\left[\frac{\xi^{-1/\eta}\eta^{-1}\gamma(\eta^{-1},\ \xi a) - |\psi^{-1}|e^{-\xi a} + |\psi_L^{-1}|}{\xi^{-1/\eta}\eta^{-1}\Gamma(\eta^{-1}) + |\psi_L^{-1}|}\right]^2, \tag{7}$$

where $\gamma(\beta,\ u)$ and $\Gamma(u)$ are the incomplete and complete Gamma functions, respectively, and $a = (|\psi^{-1}| - |\psi_L^{-1}|)^{\eta}$.

Asymptotic behavior of $K_r(S_e)$ in (7) is easy to ascertain. At $S_e = 0$ ($\psi = \psi_L$), $K_r(S_e) = 0$. At $S_e = 1$ ($\psi = 0$), $K_r(S_e) = 1$, since

$$\gamma\left(\frac{1}{\eta},\ \infty\right) = \Gamma\left(\frac{1}{\eta}\right), \tag{8a}$$

$$\lim_{\psi \to 0} \frac{1}{\psi} \exp\left[-\xi\left(\frac{1}{|\psi|} - \frac{1}{|\psi_L|}\right)^{\eta}\right] = 0. \tag{8b}$$

Between these limits, K_r increases monotonically with S_e.

3. Comparison With Experimental Data

We now test our RHC expression (7) against the experimental data from six soil types representing a wide range of soil textures and against the Brooks and Corey (5) and van Genuchten (6) models. The WRC and RHC data for the soils (Table 1) come from the catalog of *Mualem* [1974].

First, we fit the WRCs (1)–(3) to the experimental data by means of an iterative nonlinear procedure based on the Marquardt-Levenberg algorithm. In doing so, we set $\beta = 1 - \alpha^{-1}$ in (2). The fitting parameters for each model are presented in Table 2, while Table 3 demonstrates the accuracy of such a fitting by means of the root-mean-square deviation (RMSD) between the modeled and measured values,

$$\mathrm{RMSD} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}[K_{r\,\mathrm{mod}}(S_{ei}) - K_{r\,\mathrm{meas}}(S_{ei})]^2}. \tag{9}$$

Table 3. Root-Mean-Square Deviation of the Fitted WRC and the Predicted RHC Functions for the Three Models

Soil Type	WRC (1)	WRC (2)	WRC (3)	RHC (5)	RHC (6)	RHC (7)
Rubicon sandy loam	0.006	0.013	0.007	1.79	1.85	1.70
Pachappa loam	0.018	0.008	0.007	0.30	0.43	0.21
Beit Netofa clay	0.014	0.009	0.006	0.83	1.85	1.13
Pachappa fine sandy clay	0.014	0.011	0.011	0.75	1.42	1.04
Amarillo silty clay loam	0.018	0.008	0.010	0.19	0.11	0.10
Sable de Riviere	0.005	0.015	0.010	0.61	0.43	0.49

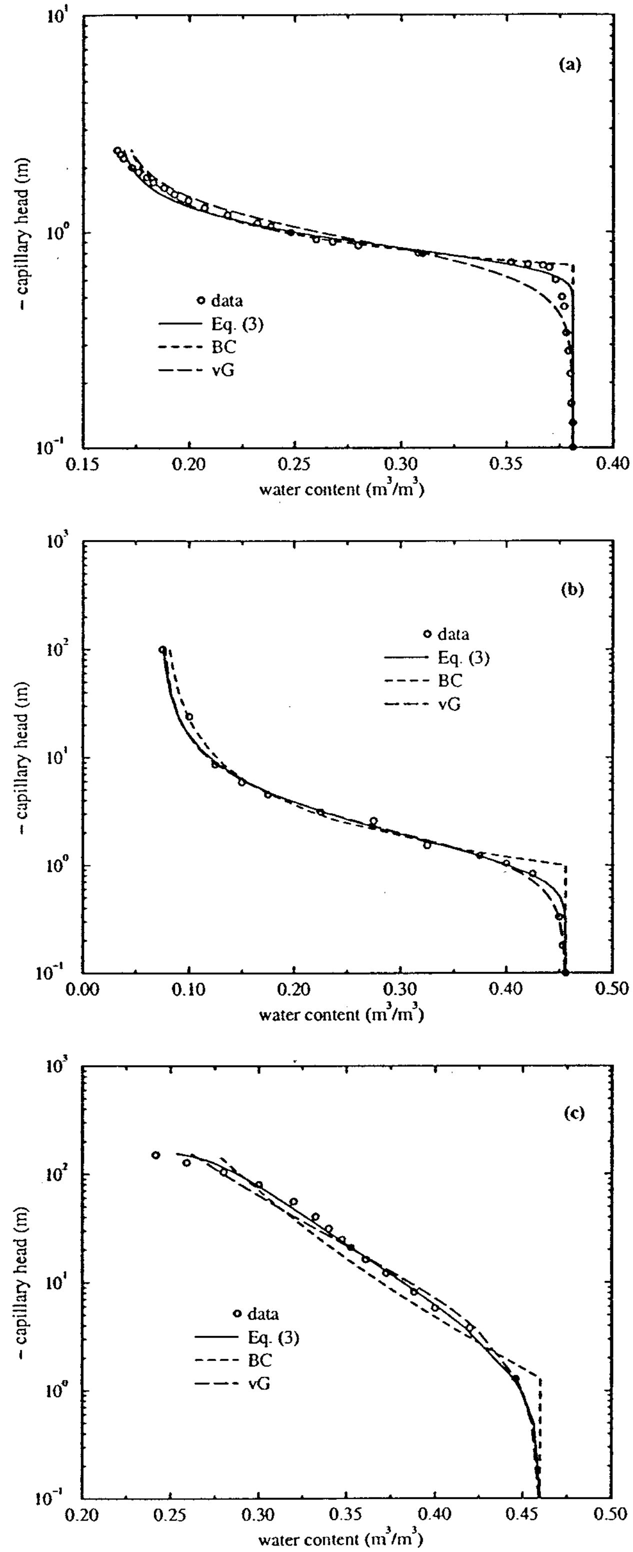

Figure 1. Comparison of the Brooks and Corey (1), van Genuchten (2), and our (3) water retention curve models (dashed, long-dashed, and solid curves, respectively) with data (circles) for (a) Rubicon sandy loam, (b) Pachappa loam, and (c) Beit Netofa clay.

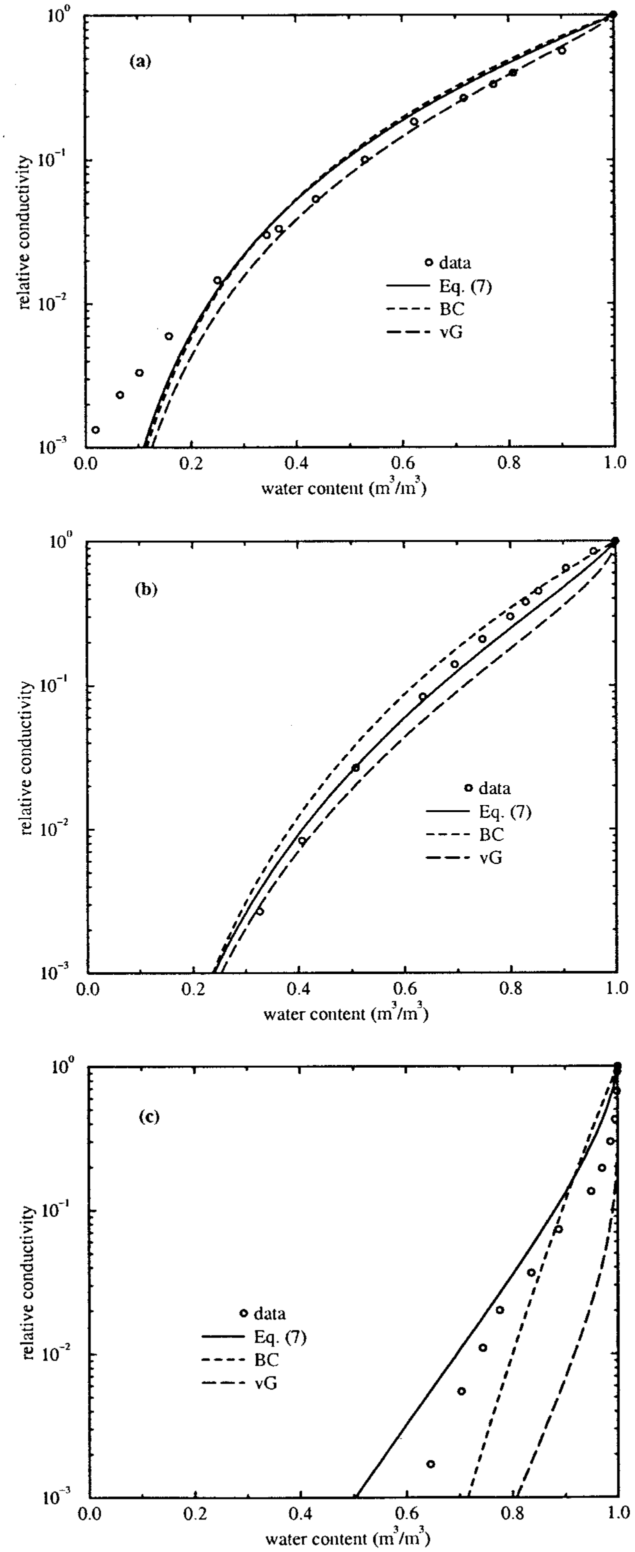

Figure 2. Comparison of the Brooks and Corey (5), van Genuchten (6), and our (7) relative hydraulic conductivity models (dashed, long-dashed, and solid curves, respectively) with data (circles) for (a) Rubicon sandy loam, (b) Pachappa loam, and (c) Beit Netofa clay.

Figures 1a–1c illustrate the ability of three WRC models to model the data for three soils. For the sandy loam (Figure 1a), WRCs based on the Brooks and Corey (1) and fragmentation-based (3) models fit the data equally well and better than the van Genuchten (2) model. Moreover, (3) provides the best representation of the $\psi(S_e)$ curve close to the air entry value. For the loam (Figure 1b), the sandy clay, and the silty clay loam soils (Table 3), (2) and (3) are virtually indistinguishable, and both fit the data better than (1). For the clay soil (Figure 1c), (2) provides the best agreement with the data, especially at low water contents. For the sandy soil (Table 3), (1) is in the best agreement with the data, although its advantage over (3) is limited to the zone of very high saturation.

We now use the parameters in Table 2 to compare the predicted RHC functions (5)–(7) with the corresponding data. Figures 2a–2c depict this comparison, and Table 3 shows the RMSD, (9) with $\ln[K_r(S_e)]$, corresponding to each curve. For the sandy loam, loam, and silty clay loam soils (Figures 2a and 2b and Table 3) our model (7) provides the best overall prediction of the data. At the same time, the pointwise performance may vary. For example, the van Genuchten model (6) predicts the RHC of the sandy loam better for $S_e > 0.5$, while our model (7) is more representative of the data for $S_e < 0.5$ (Figure 2a). For the loam, silty clay loam, and clay soils (Figures 2b and 2c and Table 3), (7) represents the RHC data best for the whole range of effective saturation. For the sandy clay soil (Table 3) all three models fail to predict the RHC data accurately, even though the Brooks and Corey model (5) performs somewhat better than the other two models. For the sandy soil (Table 3) all three RHC models are practically indistinguishable and accurately fit the data.

It thus appears that the accuracy of the WRC representation obtained via (1) and (2) does not guarantee the quality of the RHC prediction obtained via (5) and (6). Consequently, our new expression is potentially more accurate in predicting the RHC as it accounts for all the characteristics of the WRC.

4. Summary

We derived a new closed-form expression for relative hydraulic conductivity (RHC) of partially saturated soils. Our RHC can be related to the primary soil properties through a soil fragmentation model of *Assouline et al*. [1998]. This is in contrast to the widely used Brooks and Corey and van Genuchten RHCs, which are largely empirical and disconnected from basic soil properties.

We tested our RHC expression against data for a variety of soils. Overall, our RHC (7) fits the data for the six soils better than the Brooks and Corey (5) or van Genuchten (6) models. More specifically, the improvement in the predicted RHC, compared to the Brooks and Corey or van Genuchten models, depends on the soil type. For sandy soils with a step function like WRC there is practically no difference between the three models. As soil becomes heavier and its WRC becomes more sigmoid, the predictive ability of our new expression is improved.

Appendix A

Substituting (3) into (4) leads to the integrals

$$\int_0^A \frac{1}{\psi(S_e)}\, dS_e = -\xi\eta \int_{\psi_L}^{\psi(A)} \frac{1}{\psi^3}\left(\frac{1}{\psi} - \frac{1}{\psi_L}\right)^{\eta-1} \cdot \exp\left[-\xi\left(\frac{1}{\psi} - \frac{1}{\psi_L}\right)^{\eta}\right] d\psi, \tag{A1}$$

where A is either S_e or 1. The change of variables

$$x = \left(\frac{1}{|\psi|} - \frac{1}{|\psi_L|}\right)^{\eta} \tag{A2}$$

yields [*Gradshteyn and Ryzhik*, 1980, equation 3.381(1)]

$$\int_0^A \frac{1}{\psi(S_e)}\, dS_e = \xi^{-1/\eta}\gamma\left(1 + \frac{1}{\eta},\, \xi\alpha\right) - \frac{1}{\psi_L}\,(e^{-\xi\alpha} - 1), \tag{A3}$$

where $\gamma(\beta, u) = \int_0^u e^{-t}t^{\beta-1}\, dt$ is the incomplete gamma function and

$$a = \left[\frac{1}{|\psi(A)|} - \frac{1}{|\psi_L|}\right]^{\eta}. \tag{A4}$$

Since [*Gradshteyn and Ryzhik*, 1980, equation 8.356(1)]

$$\gamma\left(1 + \frac{1}{\eta},\, \xi a\right) = \frac{1}{\eta}\,\gamma\left(\frac{1}{\eta},\, \xi a\right) - (\xi a)^{1/\eta}e^{-\xi a},$$

one arrives at (7).

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