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Key Points:
- We present an efficient Bayesian updating method to detect leaks in pressurized pipes
- The efficiency is due to assimilation of data into a PDF solution of water-hammer equations
- The method allows one to identify both location and strength of a leak and accounts for uncertainty in ambient conditions

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Bayesian Update and Method of Distributions: Application to Leak Detection in Transmission Mains

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Abstract Water-hammer equations are used to describe transient flow in pipe networks. Uncertainty in model parameters, initial and boundary conditions, and location and strength of a possible leak renders deterministic predictions of this system untenable. When deployed in conjunction with pressure measurements, probabilistic solutions of the water-hammer equations serve as a tool for detecting leaks in pipes. We use the method of distributions to obtain a probability density function (PDF) for pressure head, whose dynamics are described by the stochastic water-hammer equations. This PDF provides a prior distribution for subsequent Bayesian data assimilation, in which data collected by pressure sensors are combined with this prior to obtain a posterior PDF of the leak location and size. We conduct a series of numerical experiments with uncertain initial velocity and measurement noise to ascertain the robustness and accuracy of the proposed approach. The results show the method's ability to identify the leak location and strength in a water transmission main.

1. Introduction

Transmission mains and water distribution networks are susceptible to leakage, especially as pipes age due to mechanical fatigue and corrosion. Compromised pipes raise environmental concerns and might impact public health if contaminants enter a distribution system. Perhaps more significant, leaks lead to economical losses due to the wasted resources and the cost and time required for repair. Water losses in distribution and transmission systems are estimated to be between 20% and 50% for water utilities (Brothers, 2001). This dire situation, caused by the perennial lack of funds needed to upgrade water distribution systems, puts a special premium on monitoring the reliability of pipe networks in order to localize leaks and to decrease time and cost of repairs.

Leakage is a hydraulic process that affects the pressure and flow rate in pipes. This makes tracking hydraulic characteristics and comparing them to their counterparts for intact pipes a natural way to identify leak location and size. Unfortunately, the scarcity of pressure gauges and their accuracy in pipe networks complicate this procedure at steady state, and data about the intact conditions are rarely (if ever) available. Consequently, contemporary leak detection typically relies on transient test-based techniques, in which a valve downstream of the valve or water injection devices is abruptly closed to induce unsteady flow (Brunone et al., 2008; Taghvaei et al., 2010); this procedure also creates a pressure discontinuity, which travels upstream of the pipe and carries the information about a leak to the sensors. This procedure is modeled by water-hammer equations (WHE) (Chaudhry, 2013; Wylie et al., 1993), hyperbolic partial differential equations obtained by cross-sectional averaging of the Navier-Stokes equations. A comprehensive review of transient test-based techniques is provided by Colombo et al. (2009). One of these techniques is the inverse transient analysis, which minimizes the difference between pressure measurements and (numerical) solutions of WHE to locate defects (Liggett & Chen, 1994; Vítkovsky et al., 2000; 2007). This method requires multiple pressure sensors. Another technique is referred to as the direct transient approach. It looks for a defect in the pressure signal passing through a sensor (Brunone, 1999; Brunone & Ferrante, 2001; Wang et al., 2002). The third technique is called the frequency domain method; it involves a periodically actuated device with pressure measurements confined to a part of a network or of a single pipe (Covas et al., 2005; Lee et al., 2005; Mpesha et al., 2001). The latter techniques work better in transmission mains, which can be considered as

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a closed system that holds and contains the transient pressure waves. On the other hand, water distribution networks are open systems with many branches that disperse and dampen the transient pressure waves (Meniconi et al., 2015).

These and other similar techniques assume the model parameters and operating conditions to be known with certainty (deterministic). This is seldom the case in water distribution networks that suffer from variable/uncertain demand levels, build ups on pipes, and so forth. The probabilistic framework, which equates uncertainty with randomness, provides a natural way of dealing with such complications. Some of the probabilistic approaches used to account for uncertainty in WHE are Monte Carlo simulations (MCS) (Duan, 2015; Zhang et al., 2011), polynomial chaos expansions (Sattar & El-Beltagy, 2016), and the method of distributions (Alawadhi et al., 2018). Instead of giving a single prediction of fluid pressure, these methods yield its probability density function (PDF), which can be used to assign the likelihood of occurrence (probability) to a particular prediction. It can also be used for Bayesian updating to assimilate the pressure data into model predictions, facilitating detection of leaks and estimation of their strength.

Among various data assimilation techniques, which include variational methods (Bannister, 2017), stochastic successive linear estimator (Massari et al., 2013; 2014), and different variants of Kalman filter (Ye & Fenner, 2010), we select Bayesian updating (Wikle & Berliner, 2007) because it can handle nonlinear non-Gaussian systems. Rougier and Goldstein (2001) used Bayesian analysis for pipelines with uncertain characteristics and showed that this method refines the belief about the pressure and flow. Despite its strengths, Bayesian data assimilation is notoriously computationally expensive, in part because it requires a large number of MC realizations to estimate a system state's PDF. We accelerate this step by deploying the method of distributions, which provides the prior PDF of pressure at the fraction of the computational cost of MCS (Alawadhi et al., 2018). The proposed method is used below to identify a leak location and its strength in a transmission main (single pipe).

Our paper is organized as follows. Section 2 contains a problem formulation and introduces WHE with random inputs. Its solution, given in terms of the PDF of fluid pressure in a pipe, is presented in section 3. This solution serves as a prior for Bayesian data assimilation, which is used in section 4 to locate a leak. Results of our numerical experiments are reported in section 5. Major conclusions drawn from this study are summarized in section 6.

2. Problem Formulation

Following the standard practice, we consider a reservoir-pipe-valve system (Figure 1), which represents a transmission main as a pipe of length L and diameter D. The pipe is equipped with a sensor located at $x = x^\star$ and a shutoff valve at the pipe outlet $x = L$. A water-hammer test consists of instantaneous shutoff of the downstream valve at time $t = 0$, which creates a pressure wave traveling upstream, and monitoring the pressure response at the sensor. (The term "instantaneous" should be considered as a mathematical abstraction, especially when a pipe's diameter is large.) The signal carried toward the sensor is used to identify leak location, x_{leak}, and its leak effective area (strength), $C_L A_L$. Prior to the shutoff, the flow is assumed to be steady.

For a horizontal elastic pipe, this test is described by WHE (Chaudhry, 2013; Wylie et al., 1993), which predicts the response of cross-sectionally averaged velocity $u(x, t)$ and pressure head $h(x, t)$,

$$\frac{\partial h}{\partial t} + \frac{a^2}{g}\frac{\partial u}{\partial x} = \frac{Q_{\text{leak}}}{A}\delta(x - x_{\text{leak}}), \tag{1a}$$

$$\frac{\partial u}{\partial t} + g\frac{\partial h}{\partial x} = k|u|u, \qquad k = -\frac{f}{2D}, \tag{1b}$$

where a is the wave speed; g is the gravitational acceleration constant; $A = \pi D^2/4$ is the cross-sectional area of the pipe; f is Darcy-Weisbach friction factor; x_{leak} and Q_{leak} are the unknown location and intensity of the leak, respectively; and $\delta(\cdot)$ is the Dirac delta function used to represent the leak as a point source/sink. This simplified model is based on a quasi-steady state approximation of the wall shear stress, but the method proposed below can also handle unsteady friction formulations (Meniconi et al., 2014).

Boundary conditions for (1) after instantaneous complete closure of the outlet valve are

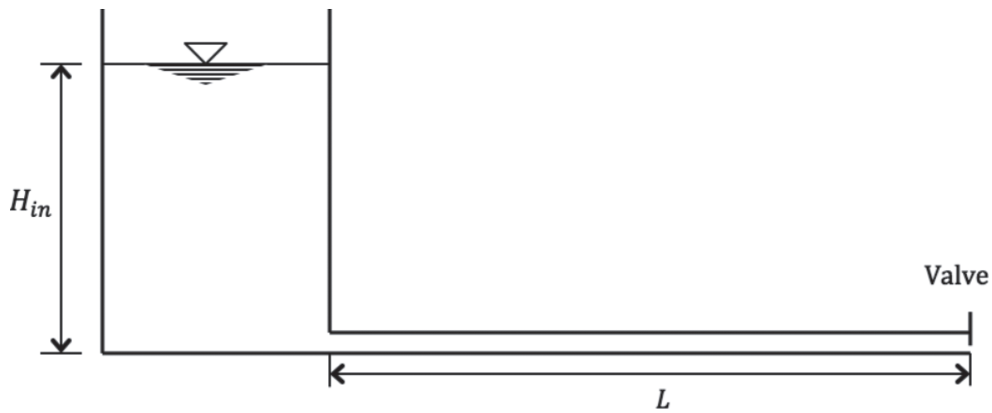

Figure 1. A reservoir-pipeline-valve system.

$$h(x = 0, t) = H_{\text{in}} \qquad \text{and} \qquad u(x = L, t) = 0, \tag{2a}$$

where H_{in} is the prescribed pressure head at the pipe inlet. The initial conditions are obtained from the steady-state conditions before the valve shuts off. The initial velocity u_0 is uniform along the pipe, while the corresponding initial pressure head h_0 is spatially varying and related to u_0 by the steady-state WHE,

$$h_0(x; u_0) = \frac{k}{g}|u_0|u_0 x + H_{\text{in}}. \tag{2b}$$

Since $\delta(x - x_{\text{leak}}) = 0$ for $x \neq x_{\text{leak}}$, equation (1) can be replaced with their homogeneous counterparts defined on the subdomains $0 < x < x_{\text{leak}}$ and $x_{\text{leak}} < x < L$. Solutions of these homogeneous equations on the two subdomains are coupled by the interfacial conditions at $x = x_{\text{leak}}$. One of these conditions is pressure continuity, $h(x_{\text{leak}}^-, t) = h(x_{\text{leak}}^+, t)$, where the subscripts $^-$ and $^+$ indicate the limits of $h(x, t)$ as $x \to x_{\text{leak}}$ from the left and the right of x_{leak}, respectively. The second condition stems from mass conservation (Brunone, 1999),

$$Q^- = Q^+ + Q_{\text{leak}}, \qquad Q_{\text{leak}}(t) = C_L A_L \sqrt{2gh(x = x_{\text{leak}}, t)}, \tag{2c}$$

and relates the flow rates upstream, $Q^- \equiv Q(x_{\text{leak}}^-, t)$, and downstream, $Q^+ \equiv Q(x_{\text{leak}}^+, t)$, of the leak. Here C_L and A_L are the discharge coefficient and the leak area, respectively.

In addition to x_{leak} and Q_{leak}, we allow the initial velocity u_0 to be uncertain. The latter uncertainty is quantified by treating u_0 as a random variable with prescribed PDF $f_{u_0}(U_0)$. In the absence of additional information, we assign to x_{leak} and $C_L A_L$ the uniform (uninformative) prior PDFs on the intervals $(0, L)$ and $[\mathcal{A}_{min}, \mathcal{A}_{max}]$, respectively:

$$x_{\text{leak}} = \mathcal{U}(0, L) \quad \text{and} \quad C_L A_L = \mathcal{U}[\mathcal{A}_{min}, \mathcal{A}_{max}].$$

We also account for measurement errors affecting the pressure sensor readings $h_{\text{obs}}(t)$,

$$h_{\text{obs}}(t) = h(x^\star, t) + \xi(t), \tag{3}$$

where $\xi(t)$ is the zero-mean Gaussian white noise with variance σ_ξ^2, that is, $\mathbb{E}[\xi(t)] = 0$ and $\mathbb{E}[\xi(t_1)\xi(t_2)] = \sigma_\xi^2 \delta(t_1 - t_2)$. This leads to a Gaussian PDF $f_{h_{\text{obs}}|h, x_{\text{leak}}}$, which accounts for the measurement error model in ((4)),

$$f_{h_{\text{obs}}|h, x_{\text{leak}}} = \frac{1}{\sqrt{2\pi\sigma_\xi^2}} \exp\left[-\frac{(H - h_{\text{obs}})^2}{2\sigma_\xi^2}\right]. \tag{4}$$

A goal of Bayesian updating is to refine the uninformed estimates of x_{leak} and $C_L A_L$ by combining the probabilistic predictions provided by (1) and (2) with observations ((4)). The PDF of $h(x, t)$, the solution of (1) and

(2), is computed in section 3. Our strategy for Bayesian updating of this prior PDF with data ((4)) is detailed in section 4.

3. Method of Distributions

When applied to (1) and (2), the method of distributions (Tartakovsky & Gremaud, 2015) yields a deterministic equation for the PDF $f_h(H; x, t)$ of the pressure head $h(x, t)$ (Alawadhi et al., 2018),

$$\frac{\partial f_h}{\partial t} + \frac{\partial V f_h}{\partial H} = 0, \qquad V = -\frac{a^2}{g}\frac{\partial \bar{u}}{\partial x} + \frac{Q_{\text{leak}}}{A}\delta(x - x_{\text{leak}}) - \alpha_1(H - \bar{h}). \tag{5a}$$

This equation describes advection of a passive scalar, f_h, in the velocity field $V(h, H, t)$. The latter depends explicitly on the mean flow velocity $\bar{u}(x, t)$ and the mean pressure head $\bar{h}(x, t)$, and implicitly on the pressure head variance $\sigma_h^2(x, t)$, through the closure variable α_1 that is given by

$$\alpha_1 = -\frac{C_L A_L \sqrt{2g}\,\delta\left(x - x_{\text{leak}}\right)}{2A\sqrt{\bar{h}}} + \frac{1}{2}\frac{\partial \ln \sigma_h^2}{\partial t}. \tag{5b}$$

These low-order statistics can be computed via, for example, moment-differential equations for (1) and (2). Instead, we compute them with MCS to avoid the closure approximations that underpin the derivation of moment-differential equations. This strategy is computationally more efficient than MCS estimation of f_h, since it takes significantly few MC realization to compute, with prescribed accuracy, the low moments (mean and variance) than the full PDF.

The PDF equation (5) is subject to the initial and boundary conditions that reflect the degree of certainty in the initial and boundary conditions of the physical system. These conditions are obtained by relating the output pressure head h_i ($i = 1, 2, \ldots$) to the initial velocity u_0 after the ith pressure wave discontinuity passes by the sensor point. This relation is computed as follows. From the PDF f_{u_0}, we draw N realizations of u_0, denoted by $u_0^{(1)}, \ldots u_0^{(N)}$. For each of these realizations, WHE (1) is solved numerically by means of the method of characteristic with explicit finite differences (Chaudhry, 2013) to obtain the values of the pressure heads after the ith discontinuity, $h_i = h_i(u_0)$, $i = 1, 2, \ldots$. This procedure results in an array of data $\{u_0^{(k)}, h_i^{(k)}\}_{k=1}^N$. Then, we fit a second-order polynomial, $u_0 = \alpha_i h_i^2 + \beta_i h_i + \gamma_i$, to these data, that is, find the coefficients α_i, β_i, and γ_i that minimize the mean root square error between the polynomial and the data. Once the map between u_0 and h_i is available, the PDF of the pressure head after each discontinuity is obtained as

$$f_{h_i} = \left|\frac{\mathrm{d}u_0}{\mathrm{d}h_i}\right| f_{u_0} \qquad \text{where} \qquad u_0 = \alpha_i h_i^2 + \beta_i h_i + \gamma_i. \tag{6}$$

Equations (5) and ((7)) are solved multiple times, for each node x_{leak} on the discretized interval $(0, L)$, to obtain $f_{h|x_{\text{leak}}}(H; x, t)$, the conditional PDF for the pressure head $h(x, t)$ given the leak location at $x = x_{\text{leak}}$. Since the PDF equation ((5a)) does not contain spatial derivatives, the physical space coordinate x acts as a parameter in this equation. Hence, it is sufficient to solve (5) and ((7)) only for the sensor node $x = x^\star$. The resulting conditional PDF $f_{h|x_{\text{leak}}}(H; x^\star, t)$ is used for data assimilation.

4. Data Assimilation

Data assimilation is implemented via a sequential Bayesian update, for each time step, as follows. Once the conditional PDF $f_{h|x_{\text{leak}}}(H; x^\star, t)$ is computed by solving the PDF equation, the joint PDF between h and x_{leak}, $f_{h,x_{\text{leak}}}(H, X; x^\star, t)$, is computed as

$$f_{h,x_{\text{leak}}}(H, X; x^\star, t) = f_{h|x_{\text{leak}}}(H; x^\star, t) f_{x_{\text{leak}}}(X), \tag{7}$$

where $f_{x_{\text{leak}}}(X)$ is the prior PDF of the leak location, for example, uniform PDF on $[0, L]$ to reflect the lack of prior knowledge. Bayesian updating (Wikle & Berliner, 2007) requires a statistical model for observations, $f_{h_{\text{obs}}|h,x_{\text{leak}}}$, which is obtained from sensor readings. Then, the posterior distribution $f_{h,x_{\text{leak}}|h_{\text{obs}}}(H, X; x^\star, t)$ is computed with the Bayes formula,

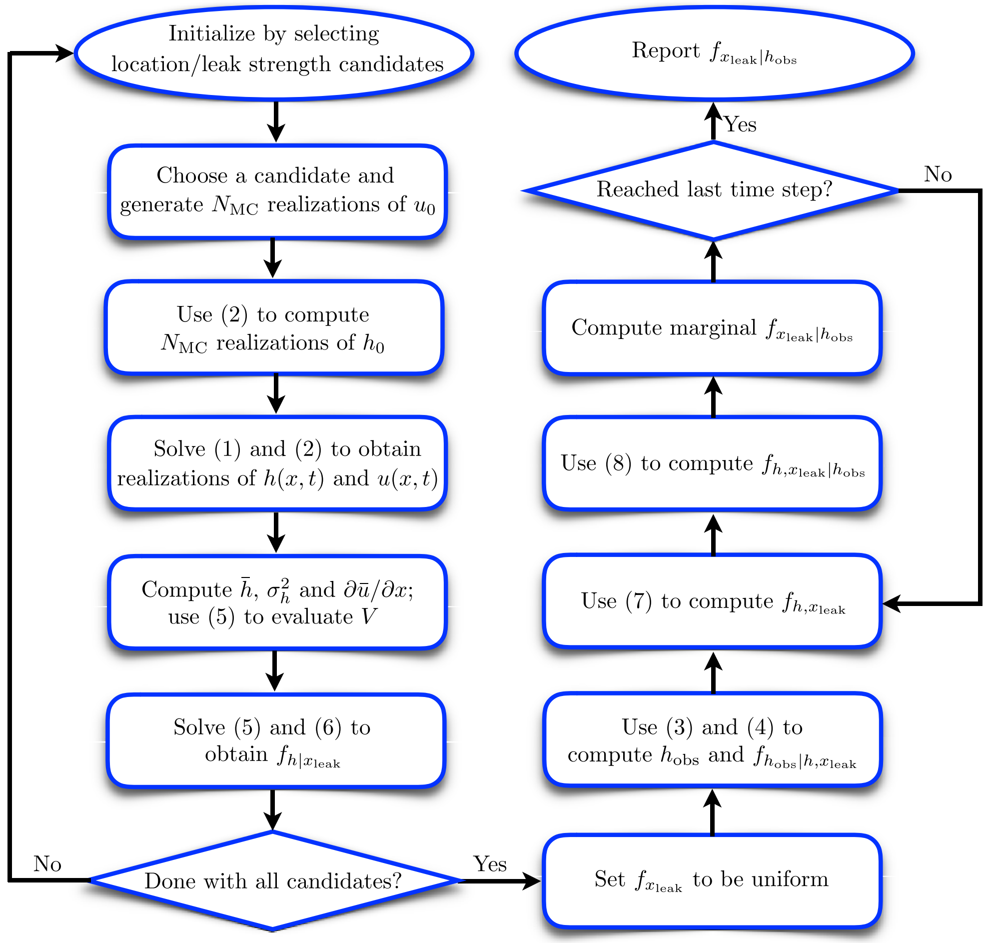

Figure 2. Workflow of the Bayesian update of the method of distribution used to locate a leak in fluid distribution mains.

$$f_{h,x_{\mathrm{leak}}|h_{\mathrm{obs}}} = \frac{f_{h_{\mathrm{obs}}|h,x_{\mathrm{leak}}} f_{h,x_{\mathrm{leak}}}}{f_{h_{\mathrm{obs}}}}, \qquad f_{h_{\mathrm{obs}}} = \int f_{h_{\mathrm{obs}}|h,x_{\mathrm{leak}}} f_{h,x_{\mathrm{leak}}} \mathrm{d}H, \tag{8}$$

where $f_{h_{\mathrm{obs}}}$ is a normalizing constant. Finally, the posterior PDF $f_{x_{\mathrm{leak}}}(X)$ is computed as the marginal of $f_{h,x_{\mathrm{leak}}|h_{\mathrm{obs}}}$, that is, by computing the integral $f_{x_{\mathrm{leak}}|h_{\mathrm{obs}}} = \int f_{h,x_{\mathrm{leak}}|h_{\mathrm{obs}}} \mathrm{d}H$. This new PDF can then be used as a prior for the next time step until the simulation horizon T is reached. The resulting PDF $f_{x_{\mathrm{leak}}|h_{\mathrm{obs}}}(X)$ is expected to be much more narrow than the initial uninformed (i.e., uniform) prior PDF. It allows one to predict the leak location with a given/required degree of certainty.

This strategy can be augmented as follows. If the leak location is known but its strength is uncertain, then $f_{x_{\mathrm{leak}}}$ is replaced with $f_{C_L A_L}$. If both leak location and its strength are unknown, then $f_{x_{\mathrm{leak}}}$ is replaced with $f_{C_L A_L, x_{\mathrm{leak}}}$, which, under the reasonable assumption of statistical independence between the leaks location and strength, becomes $f_{C_L A_L, x_{\mathrm{leak}}} = f_{C_L A_L} f_{x_{\mathrm{leak}}}$. Figure 2 provides a workflow of our method.

5. Simulation Results

In the simulations reported below, we consider a pipe of length $L = 3{,}000$ m and diameter $D = 0.5$ m. The Darcy-Weisbach friction factor is set to $f = 0.03$. A transient wave with speed $a = 1{,}200$ m/s is initiated by instantaneous complete closure of the valve at the downstream of the pipe ($x = L$). The pipe inlet ($x = 0$) has a constant pressure head of $H_{\mathrm{in}} = 25$ m. A sensor is located at the end of the pipe $x^{\star} = L = 3{,}000$ m to measure the response of the pressure head. The time required for the contact discontinuity to travel from the valve back to the inlet is $\tau = L/a = 2.5$ s. The simulation time is set to $T = 2\tau$ in order to cover two contact

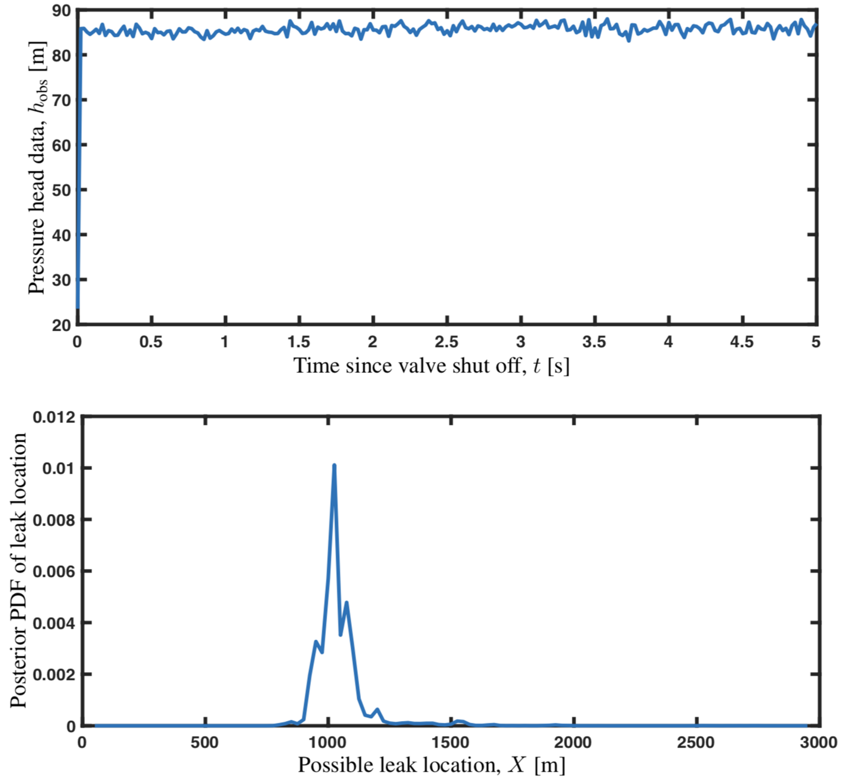

Figure 3. Scenario 1: unknown leak location x_{leak} and known leak strength $C_L A_L = 8 \cdot 10^{-5}$ m^2. The top figure exhibits the sensor's pressure head readings, $h_{\text{obs}}(t)$, for a leak located at $x_{\text{leak}} = 975$ m. The bottom figure exhibits the posterior PDF $f_{x_{\text{leak}}|h_{\text{obs}}}(X)$.

discontinuities passing through the sensor. If the leak is located at point $x = x_{\text{leak}}$, then the time needed for the contact discontinuity to reach it is $\tau_{\text{leak}} = (L - x_{\text{leak}})/a$.

Synthetic data $h_{\text{obs}}(t)$ are generated as follows. For a given leak location and strength, and for a known initial velocity chosen randomly from its distribution, WHE (1) and (2) are solved to obtain the "exact" solutions $u(x,t)$ and $h(x,t)$. The pressure time series $h^\star(t) \equiv h(x^\star, t)$ is assumed to be available only at the sensor location $x^\star$. Finally, the data $h_{\text{obs}}(t)$ are generated with ((4)) in which the variance of the zero-mean white noise $\xi(t)$ is set to $\sigma_\xi^2 = 1$ to account for measurement errors and ambient noise. This gives a Gaussian observation model with mean $h_{\text{obs}}(t)$ and variance 1.

We consider three scenarios. In the first, the leak location is unknown while its strength is certain; in the second, the leak location is certain while its strength is unknown; and in the third, both the leak location and strength are unknown, which is the general case. The first scenario applies when the leak size is known from mass balance, but the leak location is needed to start the repair. The second scenario represents a situation in which a leak's location is known, but its size needs to be determined in order to decide how fast it should be repaired. The first and second scenarios have fewer leak candidates than the third scenario, so they are computationally faster. In all cases, the pipe is subdivided into $N = 120$ equal segments. The uncertain initial velocity u_0 is modeled as a lognormal random variable such that $u_0 = 0.5 + 0.02 \exp(z)$, where z is a Gaussian random variable with mean $\mu_z = 0$ and standard deviation $\sigma_z = 0.4$. In all cases, the prior PDF $f_{h|x_{\text{leak}}}$ is computed by solving (5) and ((7)).

Scenario 1: The leak strength is known, $C_L A_L = 8 \cdot 10^{-5}$ m^2, and the unknown leak location x_{leak} is uniformly distributed in $(0, L)$. More specifically, since the simulation domain is discretized with 120 segments (121 nodes), the leak can be in any of 117 internal nodes (i.e., excluding two nodes from each edge) with equal probability $\mathbb{P} = 1/117$. Figure 3 (top figure) exhibits the sensor's pressure head readings, $h_{\text{obs}}(t)$, for a leak located at $x_{\text{leak}} = 975$ m with $u_0 = 0.518$ m/s. The leak discharge is small (less than 2% of the initial flow rate in steady-state conditions), which makes the task of leak localization challenging, especially when one accounts for ambient noise. Yet our Bayesian updating procedure is capable of

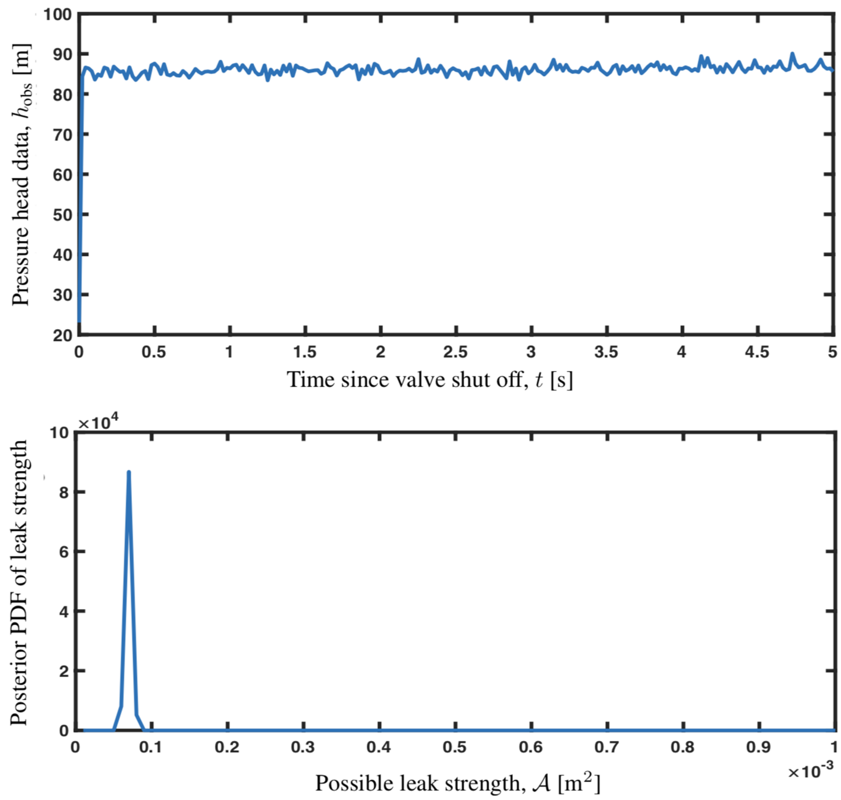

Figure 4. Scenario 2: known leak location $x_{\text{leak}} = 1{,}600$ m and unknown leak strength $C_L A_L$. The top figure exhibits the sensor's pressure head readings, $h_{\text{obs}}(t)$, for the leak strength $C_L A_L = 6 \cdot 10^{-5}$ m^2. The bottom figure exhibits the posterior PDFs $f_{C_L A_L | h_{\text{obs}}}(\mathcal{A})$.

identifying the leak location from these noisy data (Figure 3, bottom figure), because it uses the whole time series and accounts for measurement errors by representing the observed data as a distribution instead of a single value. During the update, the uninformative prior PDF $f_{x_{\text{leak}}}$ has been replaced with sharply peaked posterior PDFs $f_{x_{\text{leak}}|h_{\text{obs}}}$. The posterior PDF has the mean $\bar{x}_{\text{leak}} = 1{,}046$ m and, with the 95% confidence, places the leak between 875 and 1,229 m. Although not shown here, we found the width of the posterior PDF $f_{x_{\text{leak}}|h_{\text{obs}}}$, that is, uncertainty in the estimation of the leak location, to increase as the leak strength becomes smaller and/or the measurement noise increases. This intuitive finding serves as a consistency check for the proposed approach.

Scenario 2: The leak location is known, $x_{\text{leak}} = 1{,}600$ m, and the unknown leak strength $C_L A_L$ is uniformly distributed between 10^{-5} and 10^{-3} m^2. The pressure head response at the sensor, $h_{\text{obs}}(t)$, for the (unknown) leak strength, $C_L A_L = 6 \cdot 10^{-5}$ m^2 with $u_0 = 0.520$ m/s, is shown in Figure 4, together with the posterior PDF $f_{C_L A_L|h_{\text{obs}}}(\mathcal{A})$ used to estimate the leak strength from pressure measurement. This PDF has the mean $\overline{C_L A_L} = 7 \cdot 10^{-5}$ m^2 and, with the 95% confidence, predicts the leak strength $C_L A_L$ to lie between $5.6 \cdot 10^{-5}$ and $8 \cdot 10^{-5}$ m^2. Although not shown here, we found the performance of our method in this scenario to be sensitive to the degree of uncertainty in the initial velocity u_0. If a guessed value of u_0 is far from the mean $\bar{u}_0$, then the posterior PDF of the leak strength is centered around an erroneous value of $\overline{C_L A_L}$.

Scenario 3: Both the leak location x_{leak} and its strength $C_L A_L$ are unknown. The prior PDF $f_{x_{\text{leak}}}(X)$ is uniform on the 117 internal nodes; the prior PDF $f_{C_L A_L}(\mathcal{A})$ is uniform on the interval $[10^{-5}\ \text{m}^2, 10^{-3}\ \text{m}^2]$. The pressure head response at the sensor, $h_{\text{obs}}(t)$, for the (unknown) leak location $x_{\text{leak}} = 1{,}975$ m and the (unknown) leak strength $C_L A_L = 10^{-4}$ m^2 with $u_0 = 0.519$ m/s is shown in Figure 5. Also shown are the posterior PDFs $f_{x_{\text{leak}}|h_{\text{obs}}}(X)$ and $f_{C_L A_L|h_{\text{obs}}}(\mathcal{A})$ used to estimate, respectively, the leak location and strength from pressure measurement. Even in the presence of two sources of uncertainty, Bayesian updating is capable of accurate leak identification. The posterior PDF $f_{x_{\text{leak}}|h_{\text{obs}}}$ has the mean $\bar{x}_{\text{leak}} = 1{,}994$ m and, with the 95% confidence, places the leak between 1,925 and 2,059 m.

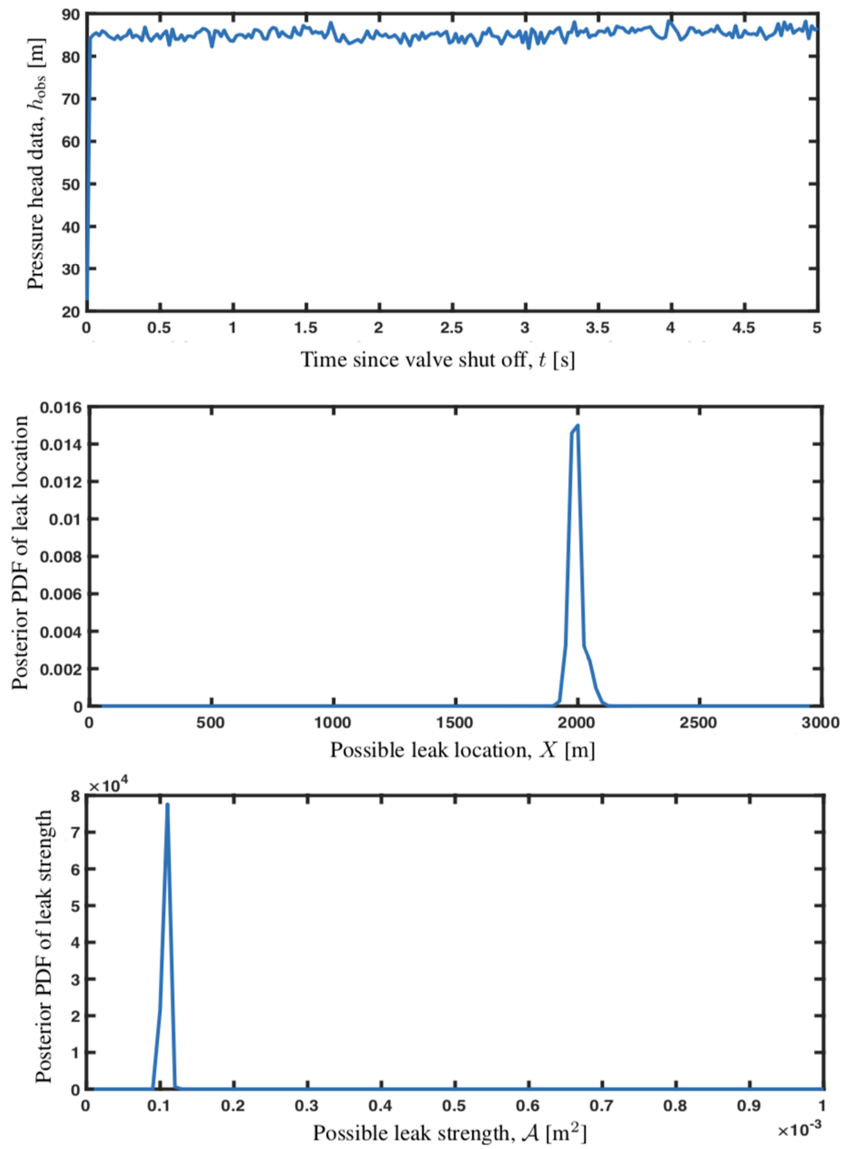

Figure 5. Scenario 3: Both leak location x_{leak} and its strength $C_L A_L$ are unknown. The top figure exhibits the sensor's pressure head readings, $h_{\text{obs}}(t)$, for the leak location $x_{\text{leak}} = 1{,}975$ m and strength $C_L A_L = 10^{-4}$ m^2. The remaining two figures depict the posterior PDFs of the leak location, $f_{x_{\text{leak}}|h_{\text{obs}}}(X)$, and leak strength, $f_{C_L A_L|h_{\text{obs}}}(\mathcal{A})$.

The posterior PDF $f_{C_L A_L|h_{\text{obs}}}$ has the mean $\overline{C_L A_L} = 1.1 \cdot 10^{-4}$ m^2 and, with the 95% confidence, predicts the leak strength $C_L A_L$ to lie between $9.4 \cdot 10^{-5}$ and $1.2 \cdot 10^{-4}$ m^2. In other words, the means of these distributions provide accurate estimates of both the location and strength of the leak, but predictive uncertainty associated with these estimators increases relative to Scenarios 1 and 2.

The results reported above show that Bayesian data assimilation, combined with the method of distributions, is a powerful tool for detection of small leaks in the presence of uncertain conditions and ambient noise. Bayesian updating produces a posterior PDFs for non-Gaussian nonlinear models. The results are accurate even for small leaks with uncertain initial velocity and errors in sensor reading.

6. Conclusions

We introduced an approach, which combines Bayesian data assimilation with the method of distributions, for leak detection in water pipes instrumented with pressure sensors. The method of distributions provides a deterministic linear equation for the PDF of pressure head, whose dynamics is described by (highly nonlinear) WHEs; this significantly reduces the computational cost, relative to MCS of the WHEs. A solution of this PDF equation serves as a prior distribution in the Bayes formula. We conducted a series of numerical experiments to demonstrate the applicability of our approach to pipe flows with uncertain initial velocity and ambient noise.

Unlike various flavors of Kalman filter, Bayesian updating does not require a system to be linear or Gaussian, that is, particular pertinent for highly nonlinear WHEs whose solutions exhibit multiple discontinuities. Our numerical experiments demonstrated how uninformed priors, which reflect the lack of knowledge about a leak's location and strength, transform themselves into sharp posterior distributions centered around the actual values of the leak's location and strength. This was done for fairly small leaks, which are characteristic of initial stages of pipe bursts. The proposed method is mainly tested to locate leaks and their strength in transmission mains. In a follow-up study we will investigate its ability to locate abnormalities in water distribution networks.

The width of the posterior PDFs, that is, uncertainty in the estimation of the leak location, increases as the leak strength becomes smaller and/or the measurement noise increases. This intuitive finding serves as a consistency check for the proposed approach.

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