CRNOGORSKA AKADEMIJA NAUKA I UMJETNOSTI
GLASNIK ODJELJENJA PRIRODNIH NAUKA, 13, 1999.

QERNOGORSKAYA AKADEMIYA NAUK I ISSKUSTV
GLASNIK OTDELENIYA ESTESTVENNYH NAUK, 13, 1999.

THE MONTENEGRIN ACADEMY OF SCIENCES AND ARTS
GLASNIK OF THE SECTION OF NATURAL SCIENCES, 13, 1999

UDK 539.319

*Vlado A. Lubarda**

Deformation Theory of Plasticity Revisited

Abstract

Deformation theory of plasticity, originally introduced for infinitesimal strains, is extended to encompass the regime of finite deformations. The framework of nonlinear continuum mechanics with logarithmic strain and its conjugate stress tensor is used to cast the formulation. A connection between deformation and flow theory of metal plasticity is discussed. Extension of theory to pressure-dependent plasticity is constructed, with an application to geomechanics. Derivations based on strain and stress decompositions are both given. Duality in constitutive structures of rate-type deformation and flow theory for fissured rocks is demonstrated.

*Department of Mechanical and Aerospace Engineering, University of California at San Diego, La Jolla, CA 92093-0411, USA

OSVRT NA DEFORMACIONU TEORIJU PLASTIČNOSTI

I z v o d

U radu je data formulacija deformacione teorije plastičnosti koja obuhvata oblast konačnih deformacija. Metodi nelinearne mehanika kontinuuma, logaritamska mjera deformacije i njen konjugovani tenzor napona su adekvatno upotrebljeni u formulaciji teorije. Veza izmedju deformacione i inkrementalne teorije plastičnosti je diskutovana na primjeru polikristalnih metala. Teorija je zatim proširena na oblast plastičnosti koja zavisi od pritiska, sa primjenom u geomehanici. Formulacije na bazi dekompozicija tenzora deformacije i napona su posebno date. Dualnost konstitutivnih struktura deformacione i inkrementalne teorije je demonstrirana na modelu stijenskih masa.

INTRODUCTION

Commonly accepted theory used in most analytical and computational studies of plastic deformation of metals and geomaterials is the so-called flow theory of plasticity (e.g., Hill, 1950,1978; Lubliner, 1992; Havner, 1992). Plastic deformation is a history dependent phenomenon, characterized by nonlinearity and irreversibility of underlining physical processes (Bell, 1968). Consequently, in flow theory of plasticity the rate of strain is expressed in terms of the rate of stress and the variables describing the current state of material. The overall response is determined incrementally by integrating the rate-type constitutive and field equations along given path of loading or deformation (Lubarda and Lee, 1981; Lubarda and Shih, 1994; Lubarda and Krajcinovic, 1995).

There has been an early theory of plasticity suggested by Hencky (1924) and Ilyushin (1947,1963), known as deformation theory of plasticity, in which total strain is given as a function of total stress. Such constitutive structure, typical for nonlinear elastic deformation, is in general inappropriate for plastic deformation, since strain there depends on both stress and stress history, and is a functional rather than a function of stress. However, deformation theory of plasticity found its application in problems of proportional or simple loading, in which all stress components increase proportionally, or nearly so, without elastic unloading ever occurring (Budiansky, 1959; Kachanov, 1971). The theory was particularly successful in bifurcation studies and determination of necking and buckling loads (Hutchinson, 1974).

Deformation theory of plasticity was originally proposed for nonlinear but infinitesimally small plastic deformation. An extension to finite strain range was discussed by Stören and Rice (1975). The purpose of this paper is to provide a formulation of the rate-type deformation theory for pressure-dependent and pressure-independent plasticity at arbitrary strains. After needed kinematic and kinetic background is introduced, the logarithmic strain and its conjugate stress are conveniently utilized to cast the formulation. Relationship between the rate-type deformation and flow theory of metal plasticity is discussed. A pressure-dependent deformation theory of plasticity is constructed and compared with a non-associative flow theory of plasticity corresponding to the Drucker-Prager yield criterion. Developments based on strain and stress decompositions are both given. Duality in the constitutive structures of deformation and flow theory for fissured rocks is demonstrated.

1 KINEMATIC PRELIMINARIES

The locations of material points of a three-dimensional body in its undeformed configuration are specified by vectors $\mathbf{X}$. Their locations in deformed configuration at time t are specified by $\mathbf{x}$, such that $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$ is one-to-one deformation mapping, assumed twice continuously differentiable. The components of $\mathbf{X}$ and $\mathbf{x}$ are material and spatial coordinates of the particle. An infinitesimal material element $\mathrm{d}\mathbf{X}$ in the undeformed configuration becomes

$$\mathrm{d}\mathbf{x} = \mathbf{F} \cdot \mathrm{d}\mathbf{X}, \quad \mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} \tag{1.1}$$

in the deformed configuration at time t. Physically possible deformation mappings have positive $\det \mathbf{F}$, hence $\mathbf{F}$ is an invertible tensor; $d\mathbf{X}$ can be recovered from $\mathrm{d}\mathbf{x}$ by inverse operation $\mathrm{d}\mathbf{X} = \mathbf{F}^{-1} \cdot \mathrm{d}\mathbf{x}$.

By polar decomposition theorem, $\mathbf{F}$ is decomposed into the product of a proper orthogonal tensor and a positive-definite symmetric tensor, such that (Truesdell and Noll, 1965)

$$\mathbf{F} = \mathbf{R} \cdot \mathbf{U} = \mathbf{V} \cdot \mathbf{R}. \tag{1.2}$$

Here, $\mathbf{U}$ is the right stretch tensor, $\mathbf{V}$ is the left stretch tensor, and $\mathbf{R}$ is the rotation tensor. Evidently, $\mathbf{V} = \mathbf{R} \cdot \mathbf{U} \cdot \mathbf{R}^T$, so that $\mathbf{U}$ and $\mathbf{V}$ share the same eigenvalues (principal stretches λ_i), while their eigenvectors are related by $\mathbf{n}_i = \mathbf{R} \cdot \mathbf{N}_i$. The right and left Cauchy-Green deformation tensors are

$$\mathbf{C} = \mathbf{F}^T \cdot \mathbf{F} = \mathbf{U}^2, \quad \mathbf{B} = \mathbf{F} \cdot \mathbf{F}^T = \mathbf{V}^2. \tag{1.3}$$

If there are three distinct principal stretches, $\mathbf{C}$ and $\mathbf{B}$ have their spectral representations (Marsden and Hughes, 1983)

$$\mathbf{C} = \sum_{i=1}^{3} \lambda_i^2 \, \mathbf{N}_i \otimes \mathbf{N}_i, \quad \mathbf{B} = \sum_{i=1}^{3} \lambda_i^2 \, \mathbf{n}_i \otimes \mathbf{n}_i. \tag{1.4}$$

2 STRAIN TENSORS

Various tensor measures of strain can be introduced. A fairly general definition of material strain measures is (Hill, 1978)

$$\mathbf{E}_{(n)} = \frac{1}{2n} \left(\mathbf{U}^{2n} - \mathbf{I}^0 \right) = \sum_{i=1}^{3} \frac{1}{2n} \left(\lambda_i^{2n} - 1 \right) \mathbf{N}_i \otimes \mathbf{N}_i, \tag{2.1}$$

where $2n$ is a positive or negative integer, and λ_i and $\mathbf{N}_i$ are the principal values and directions of $\mathbf{U}$. The unit tensor in the initial configuration is $\mathbf{I}^0$. For $n = 1$, Eq. (2.1) gives the Lagrangian or Green strain $\mathbf{E}_{(1)} = (\mathbf{U}^2 - \mathbf{I}^0)/2$, for $n = -1$ the Almansi strain $\mathbf{E}_{(-1)} = (\mathbf{I}^0 - \mathbf{U}^{-2})/2$, and for $n = 1/2$ the Biot strain $\mathbf{E}_{(1/2)} = (\mathbf{U} - \mathbf{I}^0)$. The logarithmic or Hencky strain is

$$\mathbf{E}_{(0)} = \ln \mathbf{U} = \sum_{i=1}^{3} \ln \lambda_i \, \mathbf{N}_i \otimes \mathbf{N}_i. \tag{2.2}$$

A family of spatial strain measures, corresponding to material strain measures of Eqs. (2.1) and (2.2), are

$$\boldsymbol{\mathcal{E}}_{(n)} = \frac{1}{2n} \left(\mathbf{V}^{2n} - \mathbf{I} \right) = \sum_{i=1}^{3} \frac{1}{2n} \left(\lambda_i^{2n} - 1 \right) \mathbf{n}_i \otimes \mathbf{n}_i, \tag{2.3}$$

$$\boldsymbol{\mathcal{E}}_{(0)} = \ln \mathbf{V} = \sum_{i=1}^{3} \ln \lambda_i \, \mathbf{n}_i \otimes \mathbf{n}_i. \tag{2.4}$$

The unit tensor in the deformed configuration is $\mathbf{I}$, and $\mathbf{n}_i$ are the principal directions of $\mathbf{V}$. For example, $\boldsymbol{\mathcal{E}}_{(1)} = (\mathbf{V}^2 - \mathbf{I})/2$, and $\boldsymbol{\mathcal{E}}_{(-1)} = (\mathbf{I} - \mathbf{V}^{-2})/2$, the latter being known as the Eulerian strain tensor.

Since $\mathbf{U}^{2n} = \mathbf{R}^T \cdot \mathbf{V}^{2n} \cdot \mathbf{R}$, and $\mathbf{n}_i = \mathbf{R} \cdot \mathbf{N}_i$, the material and spatial strain measures are related by

$$\mathbf{E}_{(n)} = \mathbf{R}^T \cdot \boldsymbol{\mathcal{E}}_{(n)} \cdot \mathbf{R}, \quad \mathbf{E}_{(0)} = \mathbf{R}^T \cdot \boldsymbol{\mathcal{E}}_{(0)} \cdot \mathbf{R}, \tag{2.5}$$

i.e., the former are induced from the latter by the rotation $\mathbf{R}$.

Consider a material line element $d\mathbf{x}$ in the deformed configuration at time t. If the velocity field is $\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$, the velocities of the end points of $d\mathbf{x}$ differ by

$$\mathrm{d}\mathbf{v} = \mathbf{L} \cdot \mathrm{d}\mathbf{x}, \quad \dot{\mathbf{F}} \cdot \mathbf{F}^{-1}. \tag{2.6}$$

The tensor $\mathbf{L}$ is called the velocity gradient. Its symmetric and antisymmetric parts are the rate of deformation tensor and the spin tensor

$$\mathbf{D} = \frac{1}{2} \left(\mathbf{L} + \mathbf{L}^T \right), \quad \mathbf{W} = \frac{1}{2} \left(\mathbf{L} - \mathbf{L}^T \right). \tag{2.7}$$

3 CONJUGATE STRESS TENSORS

For any material strain $\mathbf{E}_{(n)}$ of Eq. (2.1), its work conjugate stress $\mathbf{T}_{(n)}$ is defined such that the stress power per unit initial volume is

$$\mathbf{T}_{(n)} : \dot{\mathbf{E}}_{(n)} = \boldsymbol{\tau} : \mathbf{D}, \tag{3.1}$$

where $\boldsymbol{\tau} = (\det \mathbf{F})\boldsymbol{\sigma}$ is the Kirchhoff stress. The Cauchy stress is denoted by $\boldsymbol{\sigma}$. For $n = 1$, Eq. (3.1) gives

$$\mathbf{T}_{(1)} = \mathbf{F}^{-1} \cdot \boldsymbol{\tau} \cdot \mathbf{F}^{-T} = \mathbf{U}^{-1} \cdot \hat{\boldsymbol{\tau}} \cdot \mathbf{U}^{-1}. \tag{3.2}$$

The stress $\hat{\boldsymbol{\tau}} = \mathbf{R}^T \cdot \boldsymbol{\tau} \cdot \mathbf{R}$ is induced from $\boldsymbol{\tau}$ by the rotation $\mathbf{R}$. Similarly,

$$\mathbf{T}_{(-1)} = \mathbf{F}^T \cdot \boldsymbol{\tau} \cdot \mathbf{F} = \mathbf{U} \cdot \hat{\boldsymbol{\tau}} \cdot \mathbf{U}. \tag{3.3}$$

More involved is an expression for the stress conjugate to logarithmic strain, although the approximation

$$\mathbf{T}_{(0)} = \hat{\boldsymbol{\tau}} + \mathcal{O}\left(\mathbf{E}_{(n)}^2 \cdot \hat{\boldsymbol{\tau}}\right) \tag{3.4}$$

may be acceptable at moderate strains. If deformation is such that principal directions of $\mathbf{V}$ and $\boldsymbol{\tau}$ are parallel, the matrices $\mathbf{E}_{(n)}$ and $\mathbf{T}_{(n)}$ commute, and in that case $\mathbf{T}_{(0)} = \hat{\boldsymbol{\tau}}$ exactly (Hill, 1978). If principal directions of $\mathbf{U}$ remain fixed during deformation,

$$\dot{\mathbf{E}}_{(0)} = \dot{\mathbf{U}} \cdot \mathbf{U}^{-1} = \hat{\mathbf{D}}, \quad \mathbf{T}_{(0)} = \hat{\boldsymbol{\tau}}. \tag{3.5}$$

The spatial strain tensors $\boldsymbol{\mathcal{E}}_{(n)}$ in general do not have their conjugate stress tensors $\boldsymbol{\mathcal{T}}_{(n)}$ such that $\mathbf{T}_{(n)} : \dot{\mathbf{E}}_{(n)} = \boldsymbol{\mathcal{T}}_{(n)} : \dot{\boldsymbol{\mathcal{E}}}_{(n)}$. However, the spatial stress tensors conjugate to strain tensors $\boldsymbol{\mathcal{E}}_{(n)}$ can be introduced by requiring that

$$\mathbf{T}_{(n)} : \dot{\mathbf{E}}_{(n)} = \boldsymbol{\mathcal{T}}_{(n)} : \overset{\bullet}{\boldsymbol{\mathcal{E}}}_{(n)}, \tag{3.6}$$

where objective, corotational rate of strain $\boldsymbol{\mathcal{E}}_{(n)}$ is defined by

$$\overset{\bullet}{\boldsymbol{\mathcal{E}}}_{(n)} = \dot{\boldsymbol{\mathcal{E}}}_{(n)} - \boldsymbol{\omega} \cdot \boldsymbol{\mathcal{E}}_{(n)} + \boldsymbol{\mathcal{E}}_{(n)} \cdot \boldsymbol{\omega}, \quad \boldsymbol{\omega} = \dot{\mathbf{R}} \cdot \mathbf{R}^{-1}. \tag{3.7}$$

In view of the relationship $\overset{\bullet}{\boldsymbol{\mathcal{E}}}_{(n)} = \mathbf{R}\cdot\dot{\mathbf{E}}_{(n)}\cdot\mathbf{R}^T$, it follows that

$$\boldsymbol{\mathcal{T}}_{(n)} = \mathbf{R}\cdot\mathbf{T}_{(n)}\cdot\mathbf{R}^T. \tag{3.8}$$

This is the conjugate stress to spatial strains $\boldsymbol{\mathcal{E}}_{(n)}$ in the sense of Eq. (3.6).

Note that $\mathbf{R}\cdot\boldsymbol{\tau}\cdot\mathbf{R}^T$ is not the work conjugate to any strain measure, since the material stress tensor $\mathbf{T}_{(n)}$ in Eq. (3.8) cannot be equal to spatial stress tensor $\boldsymbol{\tau}$. Likewise, although $\hat{\boldsymbol{\tau}}:\hat{\mathbf{D}} = \boldsymbol{\tau}:\mathbf{D}$, the stress tensor $\hat{\boldsymbol{\tau}} = \mathbf{R}^T\cdot\boldsymbol{\tau}\cdot\mathbf{R}$ is not the work conjugate to any strain measure, because $\hat{\mathbf{D}} = \mathbf{R}^T\cdot\mathbf{D}\cdot\mathbf{R}$ is not the rate of any strain. Of course, $\boldsymbol{\tau}$ itself is not the work conjugate to any strain, because $\mathbf{D}$ is not the rate of any strain, either.

4 DEFORMATION THEORY OF PLASTICITY

Simple plasticity theory has been suggested for proportional loading and small deformation by Hencky(1924) and Ilyushin (1947,1963). A large deformation version of this theory is here presented. It is convenient to cast the formulation by using the logarithmic strain $\mathbf{E}_{(0)} = \ln\mathbf{U}$ and its conjugate stress $\mathbf{T}_{(0)}$. Assume that the loading is such that all stress components increase proportionally, i.e.

$$\mathbf{T}_{(0)} = c(t)\,\mathbf{T}^*_{(0)}, \tag{4.1}$$

where $\mathbf{T}^*_{(0)}$ is the stress tensor at instant t^*, and $c(t)$ is monotonically increasing function of t, with $c(t^*) = 1$. Evidently, principal directions of $\mathbf{T}_{(0)}$ in Eq. (4.1) remain fixed during the deformation process.

Since stress proportionally increases, with no elastic unloading taking place, it seems reasonable to assume that elastoplastic response can be described macroscopically by the constitutive structure of nonlinear elasticity, in which total strain is a function of total stress. Thus, decompose the total strain into its elastic and plastic parts,

$$\mathbf{E}_{(0)} = \mathbf{E}^{\mathrm{e}}_{(0)} + \mathbf{E}^{\mathrm{p}}_{(0)}, \tag{4.2}$$

and assume that

$$\mathbf{E}^{\mathrm{e}}_{(0)} = \frac{\partial \phi_{(0)}}{\partial \mathbf{T}_{(0)}}, \tag{4.3}$$

$$\mathbf{E}^{\mathrm{p}}_{(0)} = \varphi_{(0)} \frac{\partial f_{(0)}}{\partial \mathbf{T}_{(0)}}, \tag{4.4}$$

where $\phi_{(0)}$ is a complementary elastic strain energy per unit undeformed volume, a Legendre transform of elastic strain energy $\psi_{(0)}$,

$$\phi_{(0)} \left(\mathbf{T}_{(0)}\right) = \mathbf{T}_{(0)} : \mathbf{E}_{(0)} - \psi_{(0)} \left(\mathbf{E}_{(0)}\right). \tag{4.5}$$

Isotropic elastic behavior will be assumed, so that $\phi_{(0)} = \phi_{(0)} \left(\mathbf{T}_{(0)}\right)$ is an isotropic function of $\mathbf{T}_{(0)}$. For plastically isotropic materials, i.e. isotropic hardening, a function $f_{(0)} = f_{(0)} \left(\mathbf{T}_{(0)}\right)$ is also an isotropic function of $\mathbf{T}_{(0)}$. The scalar $\varphi_{(0)}$ is an appropriate scalar function to be determined in accord with experimental data. Clearly, principal directions of both elastic and plastic components of strain are parallel to those of $\mathbf{T}_{(0)}$, as are the principal directions of total strain $\mathbf{E}_{(0)}$. Consequently, $\mathbf{E}_{(0)}$ and $\mathbf{U}$ have their principal directions fixed during the deformation process, the matrix $\dot{\mathbf{U}}$ commutes with $\mathbf{U}$, and by Eq. (3.5)

$$\dot{\mathbf{E}}_{(0)} = \dot{\mathbf{U}} \cdot \mathbf{U}^{-1}, \quad \mathbf{T}_{(0)} = \mathbf{R}^{T} \cdot \boldsymbol{\tau} \cdot \mathbf{R}. \tag{4.6}$$

The requirement for fixed principal directions of $\mathbf{U}$ severely restricts the class of admissible deformations, precluding, for example, the case of simple shear. This is not surprising because the premise of deformation theory – proportional stressing imposes at the outset strong restrictions on the analysis.

Introducing the spatial strain

$$\boldsymbol{\mathcal{E}}_{(0)} = \mathbf{R}^{T} \cdot \mathbf{E}_{(0)} \cdot \mathbf{R}, \tag{4.7}$$

Eqs. (4.2)-(4.4) can be rewritten as

$$\boldsymbol{\mathcal{E}}_{(0)} = \boldsymbol{\mathcal{E}}^{\mathrm{e}}_{(0)} + \boldsymbol{\mathcal{E}}^{\mathrm{p}}_{(0)}, \tag{4.8}$$

$$\boldsymbol{\mathcal{E}}^{\mathrm{e}}_{(0)} = \frac{\partial \phi_{(0)}}{\partial \boldsymbol{\tau}}, \tag{4.9}$$

$$\boldsymbol{\mathcal{E}}^{\mathrm{p}}_{(0)} = \varphi_{(0)} \, \frac{\partial f_{(0)}}{\partial \boldsymbol{\tau}}. \tag{4.10}$$

Although deformation theory of plasticity is total strain theory, the rate quantities are now introduced for later comparison with the flow theory of plasticity, and for application of the resulting rate-type constitutive equations approximately beyond proportional loading. This is also needed whenever the boundary value problem of finite deformation is being solved in an incremental manner. Since $\dot{\mathbf{U}} \cdot \mathbf{U}^{-1}$ is symmetric, we have

$$\mathbf{D} = \mathbf{R} \cdot \dot{\mathbf{E}}_{(0)} \cdot \mathbf{R}^{T}, \quad \mathbf{W} = \dot{\mathbf{R}} \cdot \mathbf{R}^{-1}, \tag{4.11}$$

and

$$\dot{\mathbf{T}}_{(0)} = \mathbf{R}^{T} \cdot \overset{\circ}{\boldsymbol{\tau}} \cdot \mathbf{R}, \quad \overset{\circ}{\boldsymbol{\mathcal{E}}}_{(0)} = \mathbf{D}. \tag{4.12}$$

By differentiating (4.2)-(4.4), or by applying the Jaumann derivative to (4.8)-(4.10), there follows

$$\mathbf{D} = \mathbf{D}^{\mathrm{e}} + \mathbf{D}^{\mathrm{p}}, \tag{4.13}$$

$$\mathbf{D}^{\mathrm{e}} = \mathbf{M}_{(0)} : \overset{\circ}{\boldsymbol{\tau}}, \quad \mathbf{M}_{(0)} = \frac{\partial^2 \phi_{(0)}}{\partial \boldsymbol{\tau} \otimes \partial \boldsymbol{\tau}}, \tag{4.14}$$

$$\mathbf{D}^{\mathrm{p}} = \dot{\varphi}_{(0)} \, \frac{\partial f_{(0)}}{\partial \boldsymbol{\tau}} + \varphi_{(0)} \, \frac{\partial^2 f_{(0)}}{\partial \boldsymbol{\tau} \otimes \partial \boldsymbol{\tau}} : \overset{\circ}{\boldsymbol{\tau}}. \tag{4.15}$$

Assume quadratic representation of the complementary energy

$$\phi_{(0)} = \frac{1}{2} \mathbf{M}_{(0)} :: (\boldsymbol{\tau} \otimes \boldsymbol{\tau}), \quad \mathbf{M}_{(0)} = \frac{1}{2\mu} \left(\boldsymbol{I} - \frac{\lambda}{2\mu + 3\lambda} \mathbf{I} \otimes \mathbf{I} \right), \tag{4.16}$$

where λ and μ are the Lamé elastic constants. Furthermore, let the function $f_{(0)}$ be defined by the second invariant of deviatoric part of the Kirchhoff stress,

$$f_{(0)} = \frac{1}{2}\,\boldsymbol{\tau}' : \boldsymbol{\tau}'. \tag{4.17}$$

Substituting the last two expressions in Eq. (4.15) gives

$$\mathbf{D}^{\mathrm{p}} = \dot{\varphi}_{(0)}\,\boldsymbol{\tau}' + \varphi_{(0)}\,\overset{\circ}{\boldsymbol{\tau}}{}'. \tag{4.18}$$

The deviatoric and spherical parts of the total rate of deformation tensor are accordingly

$$\mathbf{D}' = \dot{\varphi}_{(0)}\,\boldsymbol{\tau}' + \left(\frac{1}{2\mu} + \varphi_{(0)}\right)\overset{\circ}{\boldsymbol{\tau}}{}', \tag{4.19}$$

$$\mathrm{tr}\,\mathbf{D} = \frac{1}{3\kappa}\,\mathrm{tr}\,\overset{\circ}{\boldsymbol{\tau}}, \tag{4.20}$$

where $\kappa = \lambda + (2/3)\mu$ is the elastic bulk modulus.

Suppose that a nonlinear relationship $\overline{\tau} = \overline{\tau}\,(\overline{\gamma})$ between the Kirchhoff stress and the logarithmic strain is available from elastoplastic pure shear test. Let the secant and tangent moduli be defined by

$$h_{\mathrm{s}} = \frac{\overline{\tau}}{\overline{\gamma}}, \quad h_{\mathrm{t}} = \frac{\mathrm{d}\overline{\tau}}{\mathrm{d}\overline{\gamma}}, \tag{4.21}$$

and let

$$\overline{\tau} = \left(\frac{1}{2}\,\boldsymbol{\tau}' : \boldsymbol{\tau}'\right)^{1/2} = \left(\frac{1}{2}\,\mathbf{T}'_{(0)} : \mathbf{T}'_{(0)}\right)^{1/2}, \tag{4.22}$$

$$\overline{\gamma} = \left(2\,\boldsymbol{\mathcal{E}}'_{(0)} : \boldsymbol{\mathcal{E}}'_{(0)}\right)^{1/2} = \left(2\,\mathbf{E}'_{(0)} : \mathbf{E}'_{(0)}\right)^{1/2}. \tag{4.23}$$

Since from Eqs. (4.9) and (4.10)

$$\boldsymbol{\mathcal{E}}'_{(0)} = \left(\frac{1}{2\mu} + \varphi_{(0)}\right)\boldsymbol{\tau}', \tag{4.24}$$

substitution into Eq. (4.23) provides an expression for

$$\varphi_{(0)} = \frac{1}{2h_{\rm s}} - \frac{1}{2\mu}. \tag{4.25}$$

In order to derive an expression for the rate $\dot{\varphi}_{(0)}$, differentiate Eqs. (4.22) and (4.23) to obtain

$$\overline{\tau}\,\dot{\overline{\tau}} = \frac{1}{2}\,\boldsymbol{\tau}' : \overset{\circ}{\boldsymbol{\tau}}, \quad \overline{\gamma}\,\dot{\overline{\gamma}} = 2\boldsymbol{\mathcal{E}}'_{(0)} : \mathbf{D}. \tag{4.26}$$

In view of Eqs. (4.21), (4.24) and (4.25), this gives

$$\frac{1}{2}\,\boldsymbol{\tau}' : \overset{\circ}{\boldsymbol{\tau}} = 2h_{\rm s}h_{\rm t}\,\boldsymbol{\mathcal{E}}'_{(0)} : \mathbf{D}' = h_{\rm t}\,\boldsymbol{\tau}' : \mathbf{D}'. \tag{4.27}$$

When Eq. (4.19) is incorporated into Eq. (4.27), the rate is found to be

$$\dot{\varphi}_{(0)} = \frac{1}{2}\left(\frac{1}{h_{\rm t}} - \frac{1}{h_{\rm s}}\right)\frac{\boldsymbol{\tau}' : \overset{\circ}{\boldsymbol{\tau}}}{\boldsymbol{\tau}' : \boldsymbol{\tau}'}. \tag{4.28}$$

Taking Eq. (4.28) into Eq. (4.19), the deviatoric part of the total rate of deformation is

$$\mathbf{D}' = \frac{1}{2h_{\rm s}}\left[\overset{\circ}{\boldsymbol{\tau}}{}' + \left(\frac{h_{\rm s}}{h_{\rm t}} - 1\right)\frac{(\boldsymbol{\tau}' \otimes \boldsymbol{\tau}') : \overset{\circ}{\boldsymbol{\tau}}}{\boldsymbol{\tau}' : \boldsymbol{\tau}'}\right]. \tag{4.29}$$

Eq. (4.29) can be inverted to give

$$\overset{\circ}{\boldsymbol{\tau}}{}' = 2h_{\rm s}\left[\mathbf{D}' - \left(1 - \frac{h_{\rm t}}{h_{\rm s}}\right)\frac{(\boldsymbol{\tau}' \otimes \boldsymbol{\tau}') : \mathbf{D}}{\boldsymbol{\tau}' : \boldsymbol{\tau}'}\right]. \tag{4.30}$$

During initial, purely elastic stages of deformation, $h_{\rm t} = h_{\rm s} = \mu$. The onset of plasticity, beyond which Eqs. (4.29) and (4.30) apply, occurs when $\overline{\tau}$, defined by the second invariant of the deviatoric stress in Eq. (4.22), reaches the initial yield stress in shear. The resulting theory is referred to as the J_2 deformation theory of plasticity.

5 RELATIONSHIP BETWEEN DEFORMATION AND FLOW THEORY OF PLASTICITY

For proportional loading defined by Eq. (4.1) the stress rates are

$$\dot{\mathbf{T}}_{(0)} = \frac{\dot{c}}{c}\,\mathbf{T}_{(0)}, \quad \overset{\circ}{\boldsymbol{\tau}} = \frac{\dot{c}}{c}\,\boldsymbol{\tau}. \tag{5.1}$$

Consequently, from Eq. (4.28) the plastic part of the rate of deformation tensor is

$$\dot{\varphi}_{(0)} = \frac{1}{2}\left(\frac{1}{h_{\mathrm{t}}} - \frac{1}{h_{\mathrm{s}}}\right)\frac{\dot{c}}{c}, \tag{5.2}$$

while from Eq. (4.29)

$$\mathbf{D}^{\mathrm{p}} = \mathbf{D}' - \mathbf{D}^{e'} = \frac{1}{2}\left(\frac{1}{h_{\mathrm{t}}} - \frac{1}{\mu}\right)\frac{\dot{c}}{c}\,\boldsymbol{\tau}'. \tag{5.3}$$

On the other hand, in the case of flow theory of plasticity,

$$\dot{\mathbf{E}}_{(0)} = \dot{\mathbf{E}}^{\mathrm{e}}_{(0)} + \dot{\mathbf{E}}^{\mathrm{p}}_{(0)}, \tag{5.4}$$

$$\dot{\mathbf{E}}^{\mathrm{e}}_{(0)} = \mathbf{M}_{(0)} : \dot{\mathbf{T}}_{(0)}, \quad \dot{\mathbf{E}}^{\mathrm{p}}_{(0)} = \dot{\gamma}_0\,\mathbf{T}'_{(0)}. \tag{5.5}$$

The yield surface is defined by

$$\frac{1}{2}\,\mathbf{T}'_{(0)} : \mathbf{T}'_{(0)} - k^2(\vartheta) = 0, \quad \vartheta = \int_0^t \left(2\,\dot{\mathbf{E}}^{\mathrm{p}}_{(0)} : \dot{\mathbf{E}}^{\mathrm{p}}_{(0)}\right)^{1/2} \mathrm{d}t, \tag{5.6}$$

and the consistency condition gives (Lubarda, 1991,1994)

$$\dot{\gamma}_{(0)} = \frac{1}{4k^2 h^{\mathrm{p}}_{\mathrm{t}}}\,(\boldsymbol{\tau}' : \overset{\circ}{\boldsymbol{\tau}}). \tag{5.7}$$

Here, $h^{\mathrm{p}}_{\mathrm{t}} = \mathrm{d}k/\mathrm{d}\vartheta$ designates the plastic tangent modulus. Since $\mathbf{T}_{(0)} = \mathbf{R}^T \cdot \boldsymbol{\tau} \cdot \mathbf{R}$ and $\dot{\mathbf{E}}_{(0)} = \mathbf{R}^T \cdot \mathbf{D} \cdot \mathbf{R}$, the plastic part of the rate of deformation becomes

$$\mathbf{D}^{\mathrm{p}} = \frac{1}{4k^2 h^{\mathrm{p}}_{\mathrm{t}}}\left(\boldsymbol{\tau}' \otimes \boldsymbol{\tau}'\right) : \overset{\circ}{\boldsymbol{\tau}}. \tag{5.8}$$

In view of Eq. (5.1), this simplifies to

$$\mathbf{D}^{\mathrm{p}} = \dot{\gamma}_{(0)}\, \boldsymbol{\tau}' = \frac{1}{2h_{\mathrm{t}}^{\mathrm{p}}}\, \frac{\dot{c}}{c}\, \boldsymbol{\tau}'. \tag{5.9}$$

Constitutive structures (5.3) and (5.9) are in accord since

$$\frac{1}{h_{\mathrm{t}}^{\mathrm{p}}} = \frac{1}{h_{\mathrm{t}}} - \frac{1}{\mu}. \tag{5.10}$$

The last expression holds because in shear test $k = \overline{\tau}$, $\vartheta = \overline{\gamma}^{\mathrm{p}}$, and

$$\overline{\gamma}^{\mathrm{p}} = \overline{\gamma} - \overline{\gamma}^{\mathrm{e}} = \overline{\gamma} - \frac{1}{\mu}\,\overline{\tau}, \quad \frac{\mathrm{d}\overline{\gamma}^{\mathrm{p}}}{\mathrm{d}\overline{\tau}} = \frac{\mathrm{d}\overline{\gamma}}{\mathrm{d}\overline{\tau}} - \frac{1}{\mu}. \tag{5.11}$$

Also note that by (4.25), (5.2) and (5.9) there is a connection

$$\dot{\gamma}_{(0)} - \dot{\varphi}_{(0)} = \varphi_{(0)}\, \frac{\dot{c}}{c}. \tag{5.12}$$

5.1 Application of Deformation Theory Beyond Proportional Loading

If plastic secant and tangent moduli are used, related to secant and tangent moduli with respect to total strain by

$$\frac{1}{h_{\mathrm{t}}} - \frac{1}{h_{\mathrm{t}}^{\mathrm{p}}} = \frac{1}{h_{\mathrm{s}}} - \frac{1}{h_{\mathrm{s}}^{\mathrm{p}}} = \frac{1}{\mu}, \tag{5.13}$$

the plastic part of the rate of deformation can be rewritten from Eq. (4.29) as

$$\mathbf{D}^{\mathrm{p}} = \frac{1}{2h_{\mathrm{s}}^{\mathrm{p}}}\, \mathring{\boldsymbol{\tau}}' + \left(\frac{1}{2h_{\mathrm{t}}^{\mathrm{p}}} - \frac{1}{2h_{\mathrm{s}}^{\mathrm{p}}}\right) \frac{(\boldsymbol{\tau}' \otimes \boldsymbol{\tau}') : \mathring{\boldsymbol{\tau}}}{\boldsymbol{\tau}' : \boldsymbol{\tau}'}. \tag{5.14}$$

Deformation theory agrees with flow theory of plasticity only under proportional loading, since then specification of the final state of stress also specifies the stress history. For general (non-proportional) loading, more accurate and physically appropriate is the flow theory of plasticity, particularly with an accurate modeling of the yield surface and hardening behavior. Budiansky (1959), however, indicated that deformation theory can be successfully used for certain nearly proportional loading paths, as well. The rate $\overset{\circ}{\boldsymbol{\tau}}'$ in Eq. (5.14) does not then have to be codirectional with $\boldsymbol{\tau}'$. The first and third term (both proportional to $1/2h_{\mathrm{s}}^{\mathrm{p}}$) in Eq. (5.14) do not cancel each other in this case (as they do for proportional loading) , and the plastic part of the rate of deformation depends on both components of the stress rate $\overset{\circ}{\boldsymbol{\tau}}'$, one in the direction of $\boldsymbol{\tau}'$ and the other normal to it. In contrast, according to flow theory with the von Mises smooth yield surface, the component of the stress rate $\overset{\circ}{\boldsymbol{\tau}}'$ normal to $\boldsymbol{\tau}'$ does not affect the plastic part of the rate of deformation. Physical theories of plasticity (e.g., Hill, 1967) indicate that yield surface of a polycrystalline aggregate develops a vertex at its loading stress point, so that infinitesimal increments of stress in the direction normal to $\boldsymbol{\tau}'$ indeed cause further plastic flow. Since the structure of the deformation theory of plasticity under proportional loading does not use a notion of the yield surface, Eq. (5.14) can be adopted for an approximate description of the response in the case when the yield surface develops a vertex. When Eq. (5.14) is rewritten in the form

$$\mathbf{D}^{\mathrm{p}} = \frac{1}{2h_{\mathrm{s}}^{\mathrm{p}}}\left[\overset{\circ}{\boldsymbol{\tau}}' - \frac{(\boldsymbol{\tau}'\otimes\boldsymbol{\tau}') : \overset{\circ}{\boldsymbol{\tau}}}{\boldsymbol{\tau}' : \boldsymbol{\tau}'}\right] + \frac{1}{2h_{\mathrm{t}}^{\mathrm{p}}}\,\frac{(\boldsymbol{\tau}'\otimes\boldsymbol{\tau}') : \overset{\circ}{\boldsymbol{\tau}}}{\boldsymbol{\tau}' : \boldsymbol{\tau}'}, \tag{5.15}$$

the first term on the right-hand side gives the response to component of the stress increment normal to $\boldsymbol{\tau}'$. The associated plastic modulus is $h_{\mathrm{s}}^{\mathrm{p}}$. The plastic modulus associated with component of the stress increment in the direction of $\boldsymbol{\tau}'$ is $h_{\mathrm{t}}^{\mathrm{p}}$. Therefore, for continued plastic flow with small deviations from proportional loading (so that all yield segments which intersect at the vertex are active – fully active loading), Eq. (5.15) can be used to approximately account for the effects of the yield vertex. The idea was used by Rudnicki and Rice (1975) in modeling inelastic behavior of fissured rocks, as will be discussed in section 7.1. For the full range of directions of stress increment, the relationship between the rates of stress and plastic deformation is not expected to be necessarily linear, although it should be homogeneous in these rates in the absence of time-dependent (creep) effects. A corner theory that predicts continuous variation of the stiffness and allows increasingly non-proportional increments of stress is formulated by Chistoffersen and Hutchinson (1979). When applied to the analysis of necking in thin sheets under biaxial stretching, the results were in better agreement with experimental observations than those obtained from the theory with smooth yield characterization. Similar conclusions were long known in the field of elastoplastic buckling. Deformation theory predicts the buckling loads better than the flow theory with a smooth yield surface (Hutchinson, 1974).

6 PRESSURE-DEPENDENT DEFORMATION THEORY OF PLASTICITY

To include pressure dependence and allow inelastic volume changes in deformation theory of plasticity, assume that, in place of Eq. (4.4), the plastic strain is related to stress by

$$\mathbf{E}^{\mathrm{p}}_{(0)} = \varphi_{(0)} \left[\mathbf{T}'_{(0)} + \frac{2}{3}\,\beta \left(\frac{1}{2}\,\mathbf{T}'_{(0)} : \mathbf{T}'_{(0)} \right)^{1/2} \mathbf{I}^{0} \right], \tag{6.1}$$

where β is a material parameter. It follows that the deviatoric and spherical parts of the plastic rate of deformation tensor are

$$\mathbf{D}^{\mathrm{p}\,\prime} = \dot{\varphi}_{(0)}\,\boldsymbol{\tau}' + \varphi_{(0)}\,\overset{\circ}{\boldsymbol{\tau}}{}', \tag{6.2}$$

$$\operatorname{tr}\mathbf{D}^{\mathrm{p}} = 2\beta\, J_2^{1/2} \left(\dot{\varphi}_{(0)} + \varphi_{(0)}\, \frac{\boldsymbol{\tau}' : \overset{\circ}{\boldsymbol{\tau}}}{2\,J_2} \right). \tag{6.3}$$

The invariant $J_2 = (1/2)\,\boldsymbol{\tau}' : \boldsymbol{\tau}'$ is the second invariant of deviatoric part of the Kirchhoff stress.

Suppose that a nonlinear relationship $\overline{\tau} = \overline{\tau}\,(\overline{\gamma}^{\mathrm{p}})$ between the Kirchhoff stress and the plastic part of the logarithmic strain is available from the elastoplastic shear test (needed data for brittle rocks is commonly deduced from confined compression tests; Lubarda, Mastilovic and Knap, 1996a). Let the plastic secant and tangent moduli be defined by

$$h_{\mathrm{s}}^{\mathrm{p}}=\frac{\overline{\tau}}{\overline{\gamma}^{\mathrm{p}}},\quad h_{\mathrm{t}}^{\mathrm{p}}=\frac{\mathrm{d}\overline{\tau}}{\mathrm{d}\overline{\gamma}^{\mathrm{p}}},\tag{6.4}$$

and let in three-dimensional problems of overall compressive states of stress

$$\overline{\tau}=J_2^{1/2}+\frac{1}{3}\,\alpha\,\mathrm{tr}\,\boldsymbol{\tau},\tag{6.5}$$

$$\overline{\gamma}^{\,\mathrm{p}}=\left(2\boldsymbol{\mathcal{E}}_{(0)}^{\mathrm{p}}{}'::\boldsymbol{\mathcal{E}}_{(0)}^{\mathrm{p}}{}'\right)^{1/2}=2\,\varphi_{(0)}\,J_2^{1/2}.\tag{6.6}$$

The friction-type coefficient is denoted by α. Note that from Eq. (6.1), $\boldsymbol{\mathcal{E}}_{(0)}^{p'}=\varphi_{(0)}\,\boldsymbol{\tau}'$. By using the first of Eq. (6.4), therefore,

$$\varphi_{(0)}=\frac{1}{2h_{\mathrm{s}}^{\mathrm{p}}}\,\frac{\overline{\tau}}{J_2^{1/2}}.\tag{6.7}$$

In order to derive an expression for the rate $\dot{\varphi}_{(0)}$, differentiate Eqs. (6.5) and (6.6) to obtain

$$\dot{\overline{\tau}}=\frac{1}{2}\,J_2^{-1/2}(\boldsymbol{\tau}':\overset{\circ}{\boldsymbol{\tau}})+\frac{1}{3}\,\mathrm{tr}\,\overset{\circ}{\boldsymbol{\tau}},\tag{6.8}$$

$$\dot{\overline{\gamma}}^{\mathrm{p}}=2\left[\dot{\varphi}_{(0)}\,J_2^{1/2}+\frac{1}{2}\,\varphi_{(0)}\,J_2^{-1/2}(\boldsymbol{\tau}':\overset{\circ}{\boldsymbol{\tau}})\right].\tag{6.9}$$

Combining this with the second of Eq. (6.4) gives

$$\dot{\varphi}_{(0)}=\frac{1}{2}\left(\frac{1}{h_{\mathrm{t}}^{\mathrm{p}}}-\frac{1}{h_{\mathrm{s}}^{\mathrm{p}}}\,\frac{\overline{\tau}}{J_2^{1/2}}\right)\frac{\boldsymbol{\tau}':\overset{\circ}{\boldsymbol{\tau}}}{2\,J_2}+\frac{1}{2h_{\mathrm{t}}^{\mathrm{p}}}\,\frac{1}{3}\,\alpha\,\frac{\mathrm{tr}\,\overset{\circ}{\boldsymbol{\tau}}}{J_2^{1/2}}.\tag{6.10}$$

Substituting Eqs. (6.7) and (6.10) into Eqs. (6.2) and (6.3) yields

$$\begin{aligned}\mathbf{D}^{\mathrm{p}\,\prime}=&\frac{1}{2h_{\mathrm{s}}^{\mathrm{p}}}\,\frac{\overline{\tau}}{J_2^{1/2}}\,\overset{\circ}{\boldsymbol{\tau}}'+\frac{1}{2}\left(\frac{1}{h_{\mathrm{t}}^{\mathrm{p}}}-\frac{1}{h_{\mathrm{s}}^{\mathrm{p}}}\,\frac{\overline{\tau}}{J_2^{1/2}}\right)\frac{(\boldsymbol{\tau}'\otimes\boldsymbol{\tau}'):\overset{\circ}{\boldsymbol{\tau}}}{2\,J_2}\\&+\frac{1}{2h_{\mathrm{t}}^{\mathrm{p}}}\,\frac{1}{3}\,\alpha\,\frac{\mathrm{tr}\,\overset{\circ}{\boldsymbol{\tau}}}{J_2^{1/2}}\,\boldsymbol{\tau}',\end{aligned}\tag{6.11}$$

$$\operatorname{tr}\mathbf{D}^{\mathrm{p}} = \frac{\beta}{h_{\mathrm{t}}^{\mathrm{p}}}\left(\frac{\boldsymbol{\tau}' : \overset{\circ}{\boldsymbol{\tau}}}{2\,J_2^{1/2}} + \frac{1}{3}\,\alpha\operatorname{tr}\overset{\circ}{\boldsymbol{\tau}}\right). \tag{6.12}$$

If $\alpha = 0$, i.e. $\overline{\tau} = J_2^{1/2}$, Eqs. (6.11) and (6.12) reduce to

$$\mathbf{D}^{\mathrm{p}\,\prime} = \frac{1}{2h_{\mathrm{s}}^{\mathrm{p}}}\left[\overset{\circ}{\boldsymbol{\tau}}{}' + \left(\frac{h_{\mathrm{s}}^{\mathrm{p}}}{h_{\mathrm{t}}^{\mathrm{p}}} - 1\right)\frac{(\boldsymbol{\tau}'\otimes\boldsymbol{\tau}') : \overset{\circ}{\boldsymbol{\tau}}}{2\,J_2}\right], \tag{6.13}$$

$$\operatorname{tr}\mathbf{D}^{\mathrm{p}} = \frac{\beta}{2h_{\mathrm{t}}^{\mathrm{p}}}\,\frac{\boldsymbol{\tau}' : \overset{\circ}{\boldsymbol{\tau}}}{J_2^{1/2}}. \tag{6.14}$$

6.1 Non-Coaxiality Factor

It is instructive to rewrite Eq. (6.11) in an alternative form as

$$\mathbf{D}^{\mathrm{p}\,\prime} = \frac{1}{2h_{\mathrm{t}}^{\mathrm{p}}}\,\frac{\boldsymbol{\tau}'}{J_2^{1/2}}\left(\frac{\boldsymbol{\tau}' : \overset{\circ}{\boldsymbol{\tau}}}{2\,J_2^{1/2}} + \frac{1}{3}\,\alpha\operatorname{tr}\overset{\circ}{\boldsymbol{\tau}}\right) + \frac{1}{2h_{\mathrm{s}}^{\mathrm{p}}}\,\frac{\overline{\tau}}{J_2^{1/2}}\left[\overset{\circ}{\boldsymbol{\tau}}{}' - \frac{(\boldsymbol{\tau}'\otimes\boldsymbol{\tau}') : \overset{\circ}{\boldsymbol{\tau}}}{2\,J_2}\right] \tag{6.15}$$

The first part of $\mathbf{D}^{\mathrm{p}\,\prime}$ is coaxial with $\boldsymbol{\tau}'$. The second part is in the direction of the component of the stress rate $\overset{\circ}{\boldsymbol{\tau}}{}'$ that is normal to $\boldsymbol{\tau}'$. There is no work done on this part of the plastic strain rate, i.e.

$$\boldsymbol{\tau} : \mathbf{D}^{\mathrm{p}\,\prime} = \frac{1}{2h_{\mathrm{t}}^{\mathrm{p}}}\left(\boldsymbol{\tau}' : \overset{\circ}{\boldsymbol{\tau}} + \frac{2}{3}\,\alpha\,J_2^{1/2}\operatorname{tr}\overset{\circ}{\boldsymbol{\tau}}\right). \tag{6.16}$$

Observe in passing that from Eqs. (6.12) and (6.16),

$$\operatorname{tr}\mathbf{D}^{\mathrm{p}} = \beta\,\frac{\boldsymbol{\tau} : \mathbf{D}^{\mathrm{p}\,\prime}}{J_2^{1/2}}, \tag{6.17}$$

which offers a simple physical interpretation of the parameter β.

The coefficient

$$\varsigma = \frac{1}{2h_{\mathrm{s}}^{\mathrm{p}}}\,\frac{\overline{\tau}}{J_2^{1/2}} = \frac{1}{2h_{\mathrm{s}}^{\mathrm{p}}}\left(1 + \frac{1}{3}\,\alpha\,\frac{\operatorname{tr}\boldsymbol{\tau}}{J_2^{1/2}}\right) \tag{6.18}$$

in Eq. (6.15) can be interpreted as the stress-dependent non-coaxiality factor. Other definitions of this factor appeared in the literature, e.g., Nemat-Nasser (1983).

6.2 Inverse Constitutive Relations

The deviatoric and volumetric part of the total rate of deformation are obtained by adding to (6.11) and (6.12) the elastic contributions,

$$\mathbf{D}' = \left(\frac{1}{2\mu} + \frac{1}{2h_{\mathrm{s}}^{\mathrm{p}}}\,\frac{\overline{\tau}}{J_2^{1/2}}\right)\overset{\circ}{\boldsymbol{\tau}}{}' + \frac{1}{2}\left(\frac{1}{h_{\mathrm{t}}^{\mathrm{p}}} - \frac{1}{h_{\mathrm{s}}^{\mathrm{p}}}\,\frac{\overline{\tau}}{J_2^{1/2}}\right)\frac{(\boldsymbol{\tau}'\otimes\boldsymbol{\tau}'):\overset{\circ}{\boldsymbol{\tau}}}{2\,J_2} + \frac{1}{2h_{\mathrm{t}}^{\mathrm{p}}}\,\frac{1}{3}\,\alpha\,\frac{\mathrm{tr}\,\overset{\circ}{\boldsymbol{\tau}}}{J_2^{1/2}}\,\boldsymbol{\tau}', \tag{6.19}$$

$$\mathrm{tr}\,\mathbf{D} = \frac{1}{3}\left(\frac{1}{\kappa} + \frac{\alpha\beta}{h_{\mathrm{t}}^{\mathrm{p}}}\right)\mathrm{tr}\,\overset{\circ}{\boldsymbol{\tau}} + \frac{\beta}{2h_{\mathrm{t}}^{\mathrm{p}}}\,\frac{\boldsymbol{\tau}':\overset{\circ}{\boldsymbol{\tau}}}{J_2^{1/2}}. \tag{6.20}$$

The inverse relations are found to be

$$\overset{\circ}{\boldsymbol{\tau}}{}' = 2\mu\left[\frac{1}{b}\,\mathbf{D}' - \frac{a}{bc}\,\frac{(\boldsymbol{\tau}'\otimes\boldsymbol{\tau}'):\mathbf{D}}{2\,J_2} - \frac{1}{c}\,\alpha\,\frac{\kappa}{2\mu}\,\frac{\boldsymbol{\tau}'}{J_2^{1/2}}\,\mathrm{tr}\,\mathbf{D}\right], \tag{6.21}$$

$$\mathrm{tr}\,\overset{\circ}{\boldsymbol{\tau}} = \frac{3\kappa}{c}\left[\left(1 + \frac{h_{\mathrm{t}}^{\mathrm{p}}}{\mu}\right)\mathrm{tr}\,\mathbf{D} - \beta\,\frac{\boldsymbol{\tau}':\mathbf{D}}{J_2^{1/2}}\right]. \tag{6.22}$$

The introduced parameters are

$$a = 1 - \frac{h_{\mathrm{t}}^{\mathrm{p}}}{h_{\mathrm{s}}^{\mathrm{p}}}\,\frac{\overline{\tau}}{J_2^{1/2}}\left(1 + \alpha\beta\,\frac{\kappa}{h_{\mathrm{t}}^{\mathrm{p}}}\right), \quad b = 1 + \frac{\mu}{h_{\mathrm{s}}^{\mathrm{p}}}\,\frac{\overline{\tau}}{J_2^{1/2}}, \tag{6.23}$$

and

$$c = 1 + \frac{h_{\mathrm{t}}^{\mathrm{p}}}{\mu} + \alpha\beta\,\frac{\kappa}{\mu}. \tag{6.24}$$

7 Relationship to Pressure-Dependent Flow Theory of Plasticity

For geomaterials like soils and rocks, plastic deformation has its origin in pressure dependent microscopic processes and the yield condition depends on the hydrostatic component of stress. Drucker and

Prager (1952) suggested that inelastic deformation commences when the shear stress on octahedral planes overcomes cohesive and frictional resistance to sliding. The resulting yield condition is

$$f = J_2^{1/2} + \frac{1}{3}\,\alpha\, I_1 - k = 0, \tag{7.1}$$

with α as the coefficient of internal friction, and k as the yield shear strength. The first invariant of the Kirchhoff stress is $I_1 = \operatorname{tr}\boldsymbol{\tau}$, and J_2 is the second invariant of the deviatoric part of the Kirchhoff stress. Constitutive equations in which plastic part of the rate of deformation is normal to locally smooth yield surface in stress space are referred to as associative flow rules. A sufficient condition for this constitutive structure is that material obeys the Ilyushin's work postulate (Ilyushin, 1961). However, pressure-dependent dilatant materials with internal frictional effects are not well described by associative flow rules. For example, they largely overestimate inelastic volume changes in geomaterials, and in certain high-strength steels exhibiting the strength-differential effect (by which the yield strength is higher in compression than in tension). For such materials, plastic part of the rate of strain is taken to be normal to the plastic potential surface, which is distinct from the yield surface. The resulting constitutive structure is known as a non-associative flow rule. For geomaterials whose yield is governed by the Drucker-Prager yield condition, the plastic potential can be taken as

$$\pi = J_2^{1/2} + \frac{1}{3}\,\beta\, I_1 - k = 0. \tag{7.2}$$

The material parameter β is in general different from α in Eq. (7.1). Thus,

$$\mathbf{D}^{\mathrm{p}} = \dot{\gamma}\,\frac{\partial \pi}{\partial \boldsymbol{\tau}} = \dot{\gamma}\left(\frac{1}{2}\,J_2^{-1/2}\boldsymbol{\tau}' + \frac{1}{3}\,\beta\,\mathbf{I}\right). \tag{7.3}$$

The loading index $\dot{\gamma}$ is determined from the consistency condition. Assuming known the relationship $k = k(\vartheta)$ between the shear yield stress and the generalized plastic shear strain

$$\vartheta = \int_0^t \left(2\,\mathbf{D}^{\mathrm{p}\,\prime} : \mathbf{D}^{\mathrm{p}\prime}\right)^{1/2}\,\mathrm{d}t, \tag{7.4}$$

the condition $\dot{f} = 0$ gives

$$\dot{\gamma} = \frac{1}{h_{\mathrm{t}}^{\mathrm{p}}}\left(\frac{1}{2}\,J_2^{-1/2}\boldsymbol{\tau}' + \frac{1}{3}\,\alpha\,\mathbf{I}\right) : \overset{\circ}{\boldsymbol{\tau}}. \tag{7.5}$$

The plastic tangent modulus is $h_{\mathrm{t}}^{\mathrm{p}} = \mathrm{d}k/\mathrm{d}\vartheta$. Substituting Eq. (7.5) into Eq. (7.3) results in

$$\mathbf{D}^{\mathrm{p}} = \frac{1}{h_{\mathrm{t}}^{\mathrm{p}}}\left[\left(\frac{1}{2}\,J_2^{-1/2}\boldsymbol{\tau}' + \frac{1}{3}\,\beta\,\mathbf{I}\right) \otimes \left(\frac{1}{2}\,J_2^{-1/2}\boldsymbol{\tau}' + \frac{1}{3}\,\alpha\,\mathbf{I}\right)\right] : \overset{\circ}{\boldsymbol{\tau}}. \tag{7.6}$$

A physical interpretation of the parameter β is obtained by observing from Eq. (7.3) that

$$\left(2\,\mathbf{D}^{\mathrm{p}\,\prime} : \mathbf{D}^{\mathrm{p}\,\prime}\right)^{1/2} = \frac{\boldsymbol{\tau} : \mathbf{D}^{\mathrm{p}\,\prime}}{J_2^{1/2}} = \dot{\gamma}, \quad \mathrm{tr}\,\mathbf{D}^{\mathrm{p}} = \beta\,\dot{\gamma}, \tag{7.7}$$

i.e.,

$$\beta = \frac{\mathrm{tr}\,\mathbf{D}^{\mathrm{p}}}{(2\,\mathbf{D}^{\mathrm{p}\,\prime} : \mathbf{D}^{\mathrm{p}\,\prime})^{1/2}}. \tag{7.8}$$

Thus, β is the ratio of the volumetric and shear part of the plastic strain rate, which is often called the dilatancy factor (Rudnicki and Rice, 1975). Representative values of the friction coefficient and the dilatancy factor for fissured rocks indicate that $\alpha = 0.3 - -1$ and $\beta = 0.1 - -0.5$ (Lubarda, Mastilovic and Knap, 1996b). Frictional parameter and inelastic dilatancy of material actually change with progression of inelastic deformation, but are here treated as constants. For more elaborate analysis, which accounts for their variation, the paper by Nemat-Nasser and Shokooh (1980) can be consulted.

The deviatoric and spherical parts of the total rate of deformation are

$$\mathbf{D}' = \frac{\overset{\circ}{\boldsymbol{\tau}}{}'}{2\mu} + \frac{1}{2h_{\mathrm{t}}^{\mathrm{p}}}\,\frac{\boldsymbol{\tau}'}{J_2^{1/2}}\left(\frac{\boldsymbol{\tau}' : \overset{\circ}{\boldsymbol{\tau}}}{2\,J_2^{1/2}} + \frac{1}{3}\,\alpha\,\mathrm{tr}\,\overset{\circ}{\boldsymbol{\tau}}\right), \tag{7.9}$$

$$\operatorname{tr}\mathbf{D} = \frac{1}{3\kappa}\operatorname{tr}\overset{\circ}{\boldsymbol{\tau}} + \frac{\beta}{h_{\mathrm{t}}^{\mathrm{p}}}\left(\frac{\boldsymbol{\tau}' : \overset{\circ}{\boldsymbol{\tau}}}{2\,J_2^{1/2}} + \frac{1}{3}\,\alpha\operatorname{tr}\overset{\circ}{\boldsymbol{\tau}}\right). \tag{7.10}$$

These can be inverted to give the deviatoric and spherical parts of the stress rate

$$\overset{\circ}{\boldsymbol{\tau}}{}' = 2\mu\left[\mathbf{D}' - \frac{1}{c}\,\frac{(\boldsymbol{\tau}'\otimes\boldsymbol{\tau}') : \mathbf{D}}{2\,J_2} - \frac{1}{c}\,\alpha\,\frac{\kappa}{2\mu}\,\frac{\boldsymbol{\tau}'}{J_2^{1/2}}\operatorname{tr}\mathbf{D}\right], \tag{7.11}$$

$$\operatorname{tr}\overset{\circ}{\boldsymbol{\tau}} = \frac{3\kappa}{c}\left[\left(1 + \frac{h_{\mathrm{t}}^{\mathrm{p}}}{\mu}\right)\operatorname{tr}\mathbf{D} - \beta\,\frac{\boldsymbol{\tau}' : \mathbf{D}}{J_2^{1/2}}\right]. \tag{7.12}$$

The parameter c is defined in Eq. (6.24). The last expression is identical to (6.22), as expected since (6.20) and (7.10) are in concert.

If the friction coefficient α is equal to zero, Eqs. (7.11) and (7.12) reduce to

$$\overset{\circ}{\boldsymbol{\tau}}{}' = 2\mu\left[\mathbf{D}' - \frac{1}{1 + h_{\mathrm{t}}^{\mathrm{p}}/\mu}\,\frac{(\boldsymbol{\tau}'\otimes\boldsymbol{\tau}') : \mathbf{D}}{2\,J_2}\right], \tag{7.13}$$

$$\operatorname{tr}\overset{\circ}{\boldsymbol{\tau}} = 3\kappa\left(\operatorname{tr}\mathbf{D} - \frac{\beta}{1 + h_{\mathrm{t}}^{\mathrm{p}}/\mu}\,\frac{\boldsymbol{\tau}' : \mathbf{D}}{J_2^{1/2}}\right). \tag{7.14}$$

With vanishing dilatancy factor ($\beta = 0$), these coincide with the constitutive equations of isotropic hardening pressure-independent metal plasticity.

7.1 Relationship to Yield Vertex Model for Fissured Rocks

In a brittle rock, modeled to contain a collection of randomly oriented fissures, inelastic deformation results from frictional sliding on the fissure surfaces. Individual yield surface may be associated with each fissure, so that the macroscopic yield surface is the envelope of individual yield surfaces for fissures of all orientations (Rudnicki and

Rice, 1975). Continued stressing in the same direction will cause continuing sliding on (already activated) favorably oriented fissures, and will initiate sliding for a progressively greater number of orientations. After certain amount of inelastic deformation, the macroscopic yield envelope develops a vertex at the loading point. The stress increment normal to the original stress direction will initiate or continue sliding of fissure surfaces for some fissure orientations. In isotropic hardening idealization with smooth yield surface, however, a stress increment tangential to the yield surface will cause only elastic deformation, overestimating the stiffness of the response. In order to take into account the effect of the yield vertex in an approximate way, Rudnicki and Rice, (*op. cit.*) introduced a second plastic modulus h^{p}, which governs the response to part of the stress increment directed tangentially to what is taken to be the smooth yield surface through the same stress point. Since no vertex formation is associated with hydrostatic stress increments, tangential stress increments are taken to be deviatoric, and thus

$$\mathbf{D}^{\mathrm{p}\,\prime} = \frac{1}{2h_{\mathrm{t}}^{\mathrm{p}}}\,\frac{\boldsymbol{\tau}'}{J_2^{1/2}}\left(\frac{\boldsymbol{\tau}' : \overset{\circ}{\boldsymbol{\tau}}}{2\,J_2^{1/2}} + \frac{1}{3}\,\alpha\,\mathrm{tr}\,\overset{\circ}{\boldsymbol{\tau}}\right) + \frac{1}{2h^{\mathrm{p}}}\left(\overset{\circ}{\boldsymbol{\tau}}{}' - \frac{\boldsymbol{\tau}' : \overset{\circ}{\boldsymbol{\tau}}}{2\,J_2}\,\boldsymbol{\tau}'\right). \tag{7.15}$$

The dilation induced by the small tangential stress increment is assumed to be negligible, i.e.,

$$\mathrm{tr}\,\mathbf{D}^{\mathrm{p}} = \frac{\beta}{h_{\mathrm{t}}^{\mathrm{p}}}\left(\frac{\boldsymbol{\tau}' : \overset{\circ}{\boldsymbol{\tau}}}{2\,J_2^{1/2}} + \frac{1}{3}\,\alpha\,\mathrm{tr}\,\overset{\circ}{\boldsymbol{\tau}}\right). \tag{7.16}$$

Comparing Eq. (7.15) with (6.15) of the pressure-dependent deformation theory of plasticity, it is clear that the two constitutive structures are equivalent, provided that identification is made

$$h^{\mathrm{p}} = h_{\mathrm{s}}^{\mathrm{p}}\,\frac{J_2^{1/2}}{\overline{\tau}} = \frac{1}{2\varsigma}. \tag{7.17}$$

This derivation reconciles the differences left in the literature in a debate between Rudnicki (1982) and Nemat-Nasser (1982). It should also be noted that the constitutive structure in Eq. (7.15) is intended to model the response at a yield surface vertex for small deviations from proportional loading $\overset{\circ}{\boldsymbol{\tau}} \sim \boldsymbol{\tau}'$. For increasingly non-proportional stress increments, the relationship between the stress and plastic deformation rates is not expected to be necessarily linear.

The expressions for the rate of stress in terms of the rate of deformation are obtained by inversion of the expressions based on (7.10) and (7.15). The results are given by Eqs. (6.21) and (6.22), with the parameters

$$a = 1 - \frac{h_{\mathrm{t}}^{\mathrm{p}}}{h^{\mathrm{p}}} - \alpha\beta\,\frac{\kappa}{h^{\mathrm{p}}}, \quad b = 1 + \frac{\mu}{h^{\mathrm{p}}}, \tag{7.18}$$

and with c given by Eq. (6.24). In view of the connection (7.17), expressions in Eq. (7.18) are clearly in accord with (6.23). This demonstrates a duality in the constitutive structures of deformation and flow theory for the considered models of pressure-dependent plasticity.

8 DEFORMATION THEORY BASED ON STRESS DECOMPOSITION

In the flow theory of plasticity the constitutive structure can be built by either decomposing the rate of strain or the rate of stress into elastic and plastic constituents (Hill, 1978; Lubarda, 1994,1999). It is appealing to formulate the deformation theory of plasticity in a similar manner. Thus, instead of decomposing the total strain, which was done in section 4, decompose the stress tensor into its elastic and plastic part,

$$\mathbf{T}_{(0)} = \mathbf{T}_{(0)}^{\mathrm{e}} + \mathbf{T}_{(0)}^{\mathrm{p}}, \tag{8.1}$$

and assume that for isotropic pressure-independent plasticity

$$\mathbf{T}_{(0)}^{\mathrm{e}} = 2\mu\,\mathbf{E}_{(0)} + \lambda\,\mathrm{tr}\,\mathbf{E}_{(0)}\,\mathbf{I}^0, \tag{8.2}$$

$$\mathbf{T}_{(0)}^{\mathrm{p}} = -\psi_{(0)}\,\mathbf{E}'_{(0)}, \tag{8.3}$$

where $\psi_{(0)}$ is an appropriate parameter. Note that $\mathbf{T}^{\mathrm{p}}_{(0)}$ is a deviatoric tensor, so that deviatoric part of the total stress is

$$\mathbf{T}'_{(0)} = (2\mu - \psi_{(0)})\mathbf{E}'_{(0)}. \tag{8.4}$$

Since $\overset{\circ}{\boldsymbol{\tau}}{}^{\mathrm{e}\prime} = 2\mu\,\mathbf{D}'$, from (8.4) by differentiation,

$$\overset{\circ}{\boldsymbol{\tau}}{}^{\mathrm{p}} = -\dot{\psi}_{(0)}\boldsymbol{\mathcal{E}}'_{(0)} - \psi_{(0)}\mathbf{D}'. \tag{8.5}$$

Suppose that a nonlinear relationship $\overline{\gamma} = \overline{\gamma}\,(-\overline{\tau}^{\,\mathrm{p}})$ between the logarithmic strain and plastic part of the conjugate stress is available from elastoplastic pure shear test. Let the corresponding secant and tangent compliances be defined by

$$g^{\mathrm{p}}_{\mathrm{s}} = -\frac{\overline{\gamma}}{\overline{\tau}^{\,\mathrm{p}}}, \quad g^{\mathrm{p}}_{\mathrm{t}} = -\frac{\mathrm{d}\overline{\gamma}}{\mathrm{d}\overline{\tau}^{\,\mathrm{p}}}, \tag{8.6}$$

and let

$$\overline{\tau}^{\,\mathrm{p}} = -\left(\frac{1}{2}\,\boldsymbol{\tau}^{\mathrm{p}} : \boldsymbol{\tau}^{\mathrm{p}}\right)^{1/2} = -\left(\frac{1}{2}\,\mathbf{T}^{\mathrm{p}}_{(0)} : \mathbf{T}^{\mathrm{p}}_{(0)}\right)^{1/2}, \tag{8.7}$$

while $\overline{\gamma}$ is defined as in Eq. (4.23). It follows that

$$\psi_{(0)} = \frac{2}{g^{\mathrm{p}}_{\mathrm{s}}}, \tag{8.8}$$

and

$$\dot{\psi}_{(0)} = 2\left(\frac{1}{g^{\mathrm{p}}_{\mathrm{t}}} - \frac{1}{g^{\mathrm{p}}_{\mathrm{s}}}\right)\frac{\boldsymbol{\mathcal{E}}'_{(0)} : \mathbf{D}}{\boldsymbol{\mathcal{E}}'_{(0)} : \boldsymbol{\mathcal{E}}'_{(0)}}. \tag{8.9}$$

Substituting Eq. (8.9) into Eq. (8.5), the plastic part of the Jaumann rate of the Kirchhoff stress becomes

$$\overset{\circ}{\boldsymbol{\tau}}{}^{\mathrm{p}} = -\frac{2}{g^{\mathrm{p}}_{\mathrm{s}}}\left[\mathbf{D}' + \left(\frac{g^{\mathrm{p}}_{\mathrm{s}}}{g^{\mathrm{p}}_{\mathrm{t}}} - 1\right)\frac{\left(\boldsymbol{\mathcal{E}}'_{(0)} \otimes \boldsymbol{\mathcal{E}}'_{(0)}\right) : \mathbf{D}}{\boldsymbol{\mathcal{E}}'_{(0)} : \boldsymbol{\mathcal{E}}'_{(0)}}\right]. \tag{8.10}$$

By adding the elastic contribution, the deviatoric part of the Jaumann rate of stress is

$$\overset{\circ}{\boldsymbol{\tau}}{}' = 2\left[\left(\mu - \frac{1}{g_{\mathrm{s}}^{\mathrm{p}}}\right)\mathbf{D}' - \left(\frac{1}{g_{\mathrm{t}}^{\mathrm{p}}} - \frac{1}{g_{\mathrm{s}}^{\mathrm{p}}}\right)\frac{\left(\boldsymbol{\mathcal{E}}'_{(0)} \otimes \boldsymbol{\mathcal{E}}'_{(0)}\right) : \mathbf{D}}{\boldsymbol{\mathcal{E}}'_{(0)} : \boldsymbol{\mathcal{E}}'_{(0)}}\right]. \tag{8.11}$$

This constitutive structure is in agreement with (4.30), because $\overline{\tau} = \mu\overline{\gamma} + \overline{\tau}^{\,\mathrm{p}}$, and

$$h_{\mathrm{s}} = \mu - \frac{1}{g_{\mathrm{s}}^{\mathrm{p}}}, \quad h_{\mathrm{t}} = \mu - \frac{1}{g_{\mathrm{t}}^{\mathrm{p}}}. \tag{8.12}$$

It is also noted that the parameter $\psi_{(0)}$ is related to parameter $\varphi_{(0)}$ of section 4 by

$$2\mu\,\varphi_{(0)} = \frac{\psi_{(0)}}{2\mu - \psi_{(0)}}. \tag{8.13}$$

In the case of pressure-dependent plasticity, we can take the plastic part of the stress to be related to strain according to

$$\mathbf{T}_{(0)}^{\mathrm{p}} = -\psi_{(0)}\left[\mathbf{E}'_{(0)} + \frac{1}{3}\,\beta^*\left(2\,\mathbf{E}'_{(0)} : \mathbf{E}'_{(0)}\right)^{1/2}\mathbf{I}^0\right], \tag{8.14}$$

where β^* is a new material parameter. Furthermore, define

$$\overline{\gamma} = \left(2\,\mathbf{E}'_{(0)} : \mathbf{E}'_{(0)}\right)^{1/2} + \frac{1}{3}\,\alpha^*\,\mathrm{tr}\,\mathbf{E}_{(0)}, \tag{8.15}$$

$$\overline{\tau}^{\mathrm{p}\,\prime} = -\left(\frac{1}{2}\,\mathbf{T}_{(0)}^{\mathrm{p}\;\prime} : \mathbf{T}_{(0)}^{\mathrm{p}\;\prime}\right)^{1/2}, \tag{8.16}$$

and assume known the relationship $\overline{\gamma} = \overline{\gamma}\,(-\overline{\tau}^{\mathrm{p}\,\prime})$. The friction-type coefficient is denoted by α^*. It is easily verified that (6.5) and (8.15) cannot lead to equivalent constitutive descriptions, if α and α^* are both required to be constant (although distinct) coefficients. Having this in mind, and with the plastic secant and tangent compliances defined by

$$g_{\mathrm{s}}^{\mathrm{p}} = -\frac{\overline{\gamma}}{\overline{\tau}^{\mathrm{p}\,\prime}}, \quad g_{\mathrm{t}}^{\mathrm{p}} = -\frac{\mathrm{d}\overline{\gamma}}{\mathrm{d}\overline{\tau}^{\mathrm{p}\,\prime}}, \tag{8.17}$$

it follows that

$$\psi_{(0)} = \frac{2}{g_{\mathrm{s}}^{\mathrm{p}}} \frac{\overline{\gamma}}{4 j_2^{1/2}}, \tag{8.18}$$

and

$$\dot{\psi}_{(0)} = \left(\frac{2}{g_{\mathrm{t}}^{\mathrm{p}}} - \frac{2}{g_{\mathrm{s}}^{\mathrm{p}}} \frac{\overline{\gamma}}{4 j_2^{1/2}} \right) \frac{2 \boldsymbol{\mathcal{E}}'_{(0)} : \mathbf{D}}{j_2} + \frac{2}{g_{\mathrm{t}}^{\mathrm{p}}} \frac{1}{3} \alpha^* \frac{\operatorname{tr} \mathbf{D}}{j_2^{1/2}}. \tag{8.19}$$

The notation is used $j_2 = 2 \boldsymbol{\mathcal{E}}'_{(0)} : \boldsymbol{\mathcal{E}}'_{(0)}$. Consequently,

$$\begin{aligned} \overset{\circ}{\boldsymbol{\tau}}{}^{\mathrm{p}\,\prime} = &- \frac{2}{g_{\mathrm{t}}^{\mathrm{p}}} \frac{\boldsymbol{\mathcal{E}}'_{(0)}}{j_2^{1/2}} \left(\frac{2 \boldsymbol{\mathcal{E}}'_{(0)} : \mathbf{D}}{j_2^{1/2}} + \frac{1}{3} \alpha^* \operatorname{tr} \mathbf{D} \right) \\ &- \frac{2}{g_{\mathrm{s}}^{\mathrm{p}}} \frac{\overline{\gamma}}{4 j_2^{1/2}} \left[\mathbf{D}' - \frac{2 \left(\boldsymbol{\mathcal{E}}'_{(0)} \otimes \boldsymbol{\mathcal{E}}'_{(0)} \right) : \mathbf{D}}{j_2} \right]. \end{aligned} \tag{8.20}$$

$$\operatorname{tr} \overset{\circ}{\boldsymbol{\tau}}{}^{\mathrm{p}} = - \frac{2 \beta^*}{g_{\mathrm{t}}^{\mathrm{p}}} \left(\frac{2 \boldsymbol{\mathcal{E}}'_{(0)} : \mathbf{D}}{j_2^{1/2}} + \frac{1}{3} \alpha^* \operatorname{tr} \mathbf{D} \right). \tag{8.21}$$

These give rise to dual, but not equivalent constitutive structures to those associated with Eqs. (6.12) and (6.15). Finally, it is noted that

$$\operatorname{tr} \overset{\circ}{\boldsymbol{\tau}}{}^{\mathrm{p}} = \beta^* \frac{2 \boldsymbol{\mathcal{E}}'_{(0)} : \overset{\circ}{\boldsymbol{\tau}}{}^{\mathrm{p}\,\prime}}{j_2^{1/2}}, \tag{8.22}$$

which parallels Eq. (6.17).

References

Bell, J. F., The physics of large deformation of crystalline solids, *Springer Tracts in Natural Philosophy*, Vol. 14, Springer-Verlag, Berlin (1968).

Budiansky, B., A reassessment of deformation theories of plasticity, *J. Appl. Mech.*, Vol. 26, pp. 259–264 (1959).

Christoffersen, J. and Hutchinson, J. W., A class of phenomenological corner theories of plasticity, *J. Mech. Phys. Solids*, Vol. 27, pp. 465–487 (1979).

Drucker, D. C. and Prager, W., Soil mechanics and plastic analysis or limit design, *Q. Appl. Math.*, Vol. 10, pp. 157–165 (1952).

Hencky, H., Zur Theorie plastischer Deformationen und der hierdurch im Material hervorgerufenen Nachspannungen, *Z. Angew. Math. Mech.*, Vol. 4, pp. 323–334 (1924).

Hill, R., *The Mathematical Theory of Plasticity*, Oxford University Press, London (1950).

Hill, R., The essential structure of constitutive laws for metal composites and polycrystals, *J. Mech. Phys. Solids*, Vol. 15, pp. 79–95 (1967).

Hill, R., Aspects of invariance in solid mechanics, *Advances in Applied Mechanics*, Vol. 18, pp. 1–75 (1978).

Hutchinson, J. W., Plastic buckling, *Advances in Applied Mechanics*, Vol. 14, pp. 67–144 (1974).

Ilyushin, A. A., Theory of plasticity at simple loading of the bodies exhibiting plastic hardening, *Prikl. Mat. Mekh.*, Vol. 11, 291 (1947).

Ilyushin, A. A., On the postulate of plasticity, *Prikl. Math. Mekh.*, Vol. 25, pp. 503–507 (1961).

Ilyushin, A. A., *Plasticity. Foundations of the General Mathematical Theory*, Izd. Akad. Nauk SSSR, Moscow (1963) (in Russian).

Kachanov, L. M., *Foundations of Theory of Plasticity*, North-Holland, Amsterdam (1971).

Havner, K. S., *Finite Plastic Deformation of Crystalline Solids*, Cambridge University Press, Cambridge (1992).

Lubarda, V. A., Constitutive analysis of large elasto-plastic deformation based on the multiplicative decomposition of deformation gradient, *Int. J. Solids Struct.*, Vol. 27, pp. 885–895 (1991).

Lubarda, V. A., Elastoplastic constitutive analysis with the yield surface in strain space, *J. Mech. Phys. Solids*, Vol. 42, pp. 931–952 (1994).

Lubarda, V. A., On the partition of rate of deformation in crystal plasticity, *Int. J. Plasticity*, Vol. 15, pp. 721–736 (1999).

Lubarda, V.A. and Lee, E. H., A correct definition of elastic and plastic deformation and its computational significance, *J. Appl. Mech.*, Vol. 48, pp. 35–40 (1981).

Lubarda, V. A. and Krajcinovic, D., Some fundamental issues in rate theory of damage-elastoplasticity, *Int. J. Plasticity*, Vol. 11, pp. 763–797 (1995).

Lubarda, V. A., Mastilovic, S. and Knap, J., Brittle-ductile transition in porous rocks by cap model, *ASCE J. Engr. Mech.*, Vol. 122, pp. 633–642 (1996a).

Lubarda, V. A., Mastilovic, S. and Knap, J., Some comments on plasticity postulates and non-associative flow rules, *Int. J. Mech. Sci.*, Vol. 38, pp. 247–258 (1996b).

Lubarda, V. A. and Shih, C. F., Plastic spin and related issues in phenomenological plasticity, *J. Appl. Mech.*, Vol. 61, pp. 524–529 (1994).

Lubliner, J., *Plasticity Theory*, Macmillan Publishing Comp., New York (1990).

Marsden, J. E. and Hughes, T. J. R., *Mathematical Foundations of Elasticity*, Prentice Hall, Englewood Cliffs, New Jersey (1983).

Nemat-Nasser, S., Reply to discussion of "On finite plastic flows of compressible materials with internal friction", *Int. J. Solids Struct.*, Vol. 18, pp. 361–366 (1982).

Nemat-Nasser, S., On finite plastic flow of crystalline solids and geomaterials, *J. Appl. Mech.*, Vol. 50, pp. 1114–1126 (1983).

Nemat-Nasser, S. and Shokooh, A., On finite plastic flows of compressible materials with internal friction, *Int. J. Solids Struct.*, Vol. 16, pp. 495–514 (1980).

Rudnicki, J. W., Discussion of "On finite plastic flows of compressible materials with internal friction", *Int. J. Solids Struct.*, Vol. 18, pp. 357–360 (1982).

Rudnicki, J. W. and Rice, J. R., Conditions for the localization of deformation in pressure-sensitive dilatant materials, *J. Mech. Phys. Solids*, Vol. 23, pp. 371–394 (1975).

Stören, S. and Rice, J. R., Localized necking in thin sheets, *J. Mech. Phys. Solids*, Vol. 23, pp. 421–441 (1975).

Truesdell, C. and Noll, W., The nonlinear field theories of mechanics, *Handbuch der Physik*, Band III/3, Springer-Verlag, Berlin (1965).